THE CALCULUS OF VARIATIONS

N. I. AKHIEZER

THE CALCULUS OF VARIATIONS

N. I. AKHIEZER

Translated by ALINE H. FRINK,
Pennsylvania State University

This book presents a concise and lucid treatment of the calculus of variations, striking a nice balance between theory and application. Presented so that it can be covered in a one-semester course, the book includes necessary conditions for an extremum, various sufficient conditions, direct methods for establishing the existence of solutions of extreme problems and approximating these solutions practically, and also applications to problems in the physical sciences. The extensive collection of illustrative examples of the original text has been supplemented by the translator with lists of interesting problems. Finally, this book gives the scientist and engineer a full discussion of this very useful tool presented in a way that makes it easily accessible. The mathematician will enjoy the clarity and elegance of the development of the basic theory.

The Calculus of Variations

A Blaisdell Book in the Pure and Applied Sciences

George Springer, *University of Kansas*
CONSULTING EDITOR

BLAISDELL PUBLISHING COMPANY
A DIVISION OF RANDOM HOUSE, INC.

Naum I. Akhiezer

THE CALCULUS OF VARIATIONS

TRANSLATED FROM THE RUSSIAN BY *Aline H. Frink*

BLAISDELL PUBLISHING COMPANY
New York · London
A DIVISION OF RANDOM HOUSE

FIRST EDITION, 1962

© COPYRIGHT 1962, BY BLAISDELL PUBLISHING COMPANY,
A DIVISION OF RANDOM HOUSE, INC.

ALL RIGHTS RESERVED UNDER INTERNATIONAL AND PAN-AMERICAN COPYRIGHT CONVENTIONS. PUBLISHED IN NEW YORK, NEW YORK, AND IN LONDON, ENGLAND, BY BLAISDELL PUBLISHING COMPANY.

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 62-11834

MANUFACTURED IN THE UNITED STATES OF AMERICA

AUTHOR'S PREFACE

THESE LECTURES were given several times in the mechanics and mathematics division of the faculty of mathematics and physics at the State University of Khar'kov.

With the present programs of university instruction in mind I tried less to present a detailed treatment of all problems of the calculus of variations than to acquaint the student with the problems and the main methods of the calculus of variations, both classical and direct. This gap is partially filled by the section "Supplementary Topics and Exercises" (in short, the Appendix). For the sake of compactness classical examples of the calculus of variations are introduced there. The numbers in parentheses beside the numbers of the sections of the Appendix indicate the sections of the main text where the topic was first introduced.

In addition to the well-known treatises in the calculus of variations I partially used a lithographed course of the late N. G. Cebotarev and also various papers in the state journals.

I wish to thank B. Ya. Levin and N. S. Landkoff for reading the manuscript and for their useful suggestions.

N. I. Akhiezer

TRANSLATOR'S PREFACE

ALTHOUGH the calculus of variations goes back to the time of Euler, the choice of English textbooks for a first course in the subject is very limited. It seemed to me that a translation of Akhiezer's excellent book would be a welcome addition to the literature in English. It is suitable for advanced undergraduates as well as graduate students.

The book contains a large number of topics which are handled quite rigorously. It presents the classical theory and has a chapter on the newer direct methods. Exercises have been introduced at the end of each chapter; with the exception of the last three, these exercises and their answers were taken from another Russian book, "Calculus of Variations" by L. E. El'sgol'c, Moscow, 1958.

Aline H. Frink

CONTENTS

1. Equations of the Calculus of Variations 4

1 THE FUNDAMENTAL PROBLEM OF THE CALCULUS OF VARIATIONS 4
2 ABSOLUTE AND RELATIVE EXTREMA 5
3 THE FIRST NECESSARY CONDITION FOR EXTREMA 8
4 COROLLARIES OF THE FIRST NECESSARY CONDITION 12
5 INVARIANCE OF THE EULER EQUATION 14
6 A THEOREM OF HILBERT 21
7 THE EXISTENCE OF EXTREMALS 22
8 DEPENDENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS ON PARAMETERS 26
9 A THEOREM OF BERNSTEIN 30
10 THE CALCULUS OF VARIATIONS PROBLEM IN PARAMETRIC FORM 37

2. Theory of Fields 47

11 FIELDS FOR FUNCTIONALS IN ORDINARY FORM 47
12 CONSTRUCTION OF A FIELD 50
13 FIELDS FOR FUNCTIONALS IN PARAMETRIC FORM 54
14 THE WEIERSTRASS *E*-FUNCTION 56
15 SUFFICIENT CONDITIONS FOR A STRONG OR A WEAK MINIMUM FOR A FUNCTIONAL IN ORDINARY FORM 58
16 SUFFICIENT CONDITIONS FOR A STRONG OR A WEAK MINIMUM FOR A FUNCTIONAL IN PARAMETRIC FORM 63
17 NECESSARY CONDITIONS OF WEIERSTRASS AND LEGENDRE 64
18 THE JACOBI CONDITION 69
19 THE HAMILTON-JACOBI THEORY 81

3. Generalizations of the Fundamental Problem 91

20 VARIATIONS OF A FUNCTIONAL 91
21 VARIABLE END POINTS 93
22 VARIATIONS OF DOUBLE INTEGRALS 99
23 VARIATIONS OF FUNCTIONALS THAT DEPEND ON DERIVATIVES OF HIGHER ORDER 104
24 THE ELEMENTARY MULTIPLIER RULE; THE GRAM DETERMINANT 109
25 THE ISOPERIMETRIC PROBLEM 113
26 THE LAGRANGE PROBLEM 117

4. Direct Methods of the Calculus of Variations 127

27 THE CONCEPT OF DIRECT METHODS 127
28 THE ABSOLUTE MINIMUM FOR A FUNCTIONAL IN ORDINARY FORM 131
29 SOME AUXILIARY CONSIDERATIONS 134
30 A THEOREM ABOUT ABSOLUTE MINIMA 137
31 THE EULER EQUATION 139
32 THE CONSTRUCTION OF A MINIMIZING SEQUENCE 143
33 THE ABSOLUTE MINIMUM PROBLEM FOR A FUNCTIONAL IN PARAMETRIC FORM 148
34 PROOF OF HILBERT'S THEOREM 150
35 AN AUXILIARY PROPOSITION 151
36 PROOF OF TONELLI'S THEOREM 154

APPENDIX. Supplementary Topics and Exercises 161

The Calculus of Variations

1

Equations of the Calculus of Variations

1. THE FUNDAMENTAL PROBLEM OF THE CALCULUS OF VARIATIONS

This problem may be formulated as follows: In some region G of the space of the real variables $x, y_1, y_2, \cdots, y_n$, for arbitrary real finite values $z_1, z_2, \cdots, z_n$,* we have a real function

$$f(x, y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_n) = f(x, Y, Z).$$

It is continuous in all its arguments and Y and Z denote the vectors $\{y_1, y_2, \cdots, y_n\}$, $\{z_1, z_2, \cdots, z_n\}$. We shall consider the collection $\mathfrak{M}$ of all piecewise-smooth curves† $Y = Y(x)$ lying in the region G and joining the points $(a, a_1, \cdots, a_n) = (a, A)$ and $(b, b_1, \cdots, b_n) = (b, B)$. Along each such *admissible* comparison curve the integral

$$\int_a^b f(x, Y, Y')\, dx \tag{1}$$

has a well-defined value. We wish to find the curve or curves for which this integral has an *extremum*, that is, a *maximum* or *minimum* value.

The set of all piecewise-smooth vector functions defined in a given interval $[a, b]$ is called a *function space.* Each piecewise-smooth vector function is an element or point of this space, and

*In a more general statement of the problem, other conditions can be imposed on $z_1, z_2, \cdots, z_n$. An example appears in Section 26 of the Appendix.

†In other words, the collection of all continuous vector functions $Y(x) = \{y_1(x), y_2(x), \cdots, y_n(x)\}$ with piecewise-continuous derivatives $Y'(x) = \{y_1'(x), y_2'(x), \cdots, y_n'(x)\}$. Smooth and piecewise-smooth functions of several variables are discussed in Sections 22 and A-29.

the collection $\mathfrak{M}$ is some manifold in it. The integral (1) assigns a definite number to each element $Y = Y(x)$ of some manifold $\mathfrak{M}$ in the function space; in the terminology of functional analysis the integral (1) is a *functional* of Y on this manifold:

$$\int_a^b f(x, Y, Y')\, dx = J[Y]. \tag{2}$$

In contrast to the differential calculus, the calculus of variations is concerned with maximizing or minimizing functionals (functions of points in function spaces) rather than functions of points in finite-dimensional spaces.*

One of the first problems of the calculus of variations was proposed by John Bernoulli in 1696. It is called the "brachistochrone": Given two points (a, a_1), (b, b_1) in a vertical plane that do not lie on the same vertical line, find a curve joining them so that a particle starting from rest will traverse this curve without friction from one point to the other in the shortest possible time.

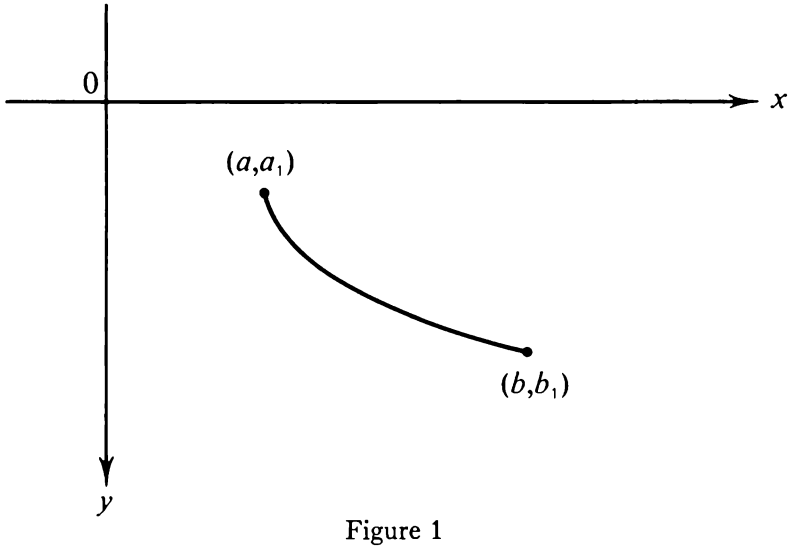

Figure 1

If we draw a pair of coordinate axes as in Figure 1, then from the formula for kinetic energy we know that

$$\tfrac{1}{2}mv^2 = mg(y - a_1).$$

*Functionals also appear in other forms, such as double integrals, elsewhere in this book.

But

$$v = \frac{ds}{dt} = \sqrt{1 + y'^2}\,\frac{dx}{dt}.$$

Hence

$$dt = \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y - a_1)}}\,dx.$$

The time of descent along the curve $y = y(x)$ is given by

$$T = J[Y] = \int_a^b \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y - a_1)}}\,dx.$$

This is the functional to be minimized in Bernoulli's problem; the region G is the half plane $y > a_1$ and one of the end points of the curve lies on the boundary of the region G.

In the general problem of the functional $J[Y]$ defined by equation (2), it is convenient to impose more conditions than are necessary. In addition to making the natural assumption of continuity for the function $f(x, Y, Y')$, we shall assume the existence and continuity of all its partial derivatives up to and including those of the third order. In some respects this last restriction is excessive.*

2. ABSOLUTE AND RELATIVE EXTREMA

An extremum in the whole collection $\mathfrak{M}$ is called an *absolute* extremum. We shall also consider *relative* extrema; to define them we must introduce the notion of *neighborhood.*

A *neighborhood of zero order*, that is, a *strong neighborhood*, of a continuous vector function $\bar{Y}(x)$ is the collection of all continuous vector functions $Y(x)$ for which the inequality

$$\|Y(x) - \bar{Y}(x)\| = \sqrt{[y_1(x) - \bar{y}_1(x)]^2 + \cdots + [y_n(x) - \bar{y}_n(x)]^2} \leq \epsilon \quad (1)$$

*Chapters 1 and 2 are devoted to the fundamental problem and develop the theory quite thoroughly. Assumptions concerning the integrand function are necessarily more stringent than those in Chapter 3 and in the Appendix where the theory is developed less fully for various generalizations of the fundamental problem.

holds for $a \leq x \leq b$; here $\epsilon > 0$ is a number characterizing the neighborhood; such a neighborhood is called an ϵ-neighborhood.

A *neighborhood of the first order*, that is, a *weak neighborhood* of a piecewise-smooth vector function $\bar{Y}(x)$ is the collection of all piecewise-smooth vector functions $Y(x)$ that satisfy inequality (1) and also satisfy the relation

$$\|Y'(x) - \bar{Y}'(x)\| \leq \epsilon$$

for every point x of the interval $[a, b]$ at which $\bar{Y}'(x)$ and $Y'(x)$ exist.

The curve $Y = \bar{Y}(x)$ may furnish an extremum for the functional only in some neighborhood of $Y = \bar{Y}(x)$ lying in $\mathfrak{M}$. In this case, the extremum is called *relative* and is designated as *strong* or *weak* according to whether the neighborhood in which it lies is strong or weak. A curve that furnishes an extremum in such a collection of comparison curves may not supply an extremum if the collection is enlarged. Hence a curve furnishing a weak relative extremum may not supply a strong relative extremum and *a fortiori* an absolute extremum. In studying weak relative extrema, the following proposition will be very useful.

Consider a smooth curve $Y = \bar{Y}(x)$ that gives the functional $J[Y]$ an extremum in the class of all admissible smooth curves in some weak neighborhood of $Y = \bar{Y}(x)$. Such a curve also gives the functional $J[Y]$ an extremum in the class of all admissible piecewise-smooth curves in the same neighborhood. In other words, this curve supplies the functional $J[Y]$ with a weak relative extremum.

To prove this proposition, we employ a process known as *rounding off the corners*. We begin by assuming that the extremum is a minimum, and then choose an admissible piecewise-smooth curve $Y = \tilde{Y}(x)$ that satisfies the condition

$$J[\tilde{Y}] = J[\bar{Y}] - h \qquad (h > 0)$$

in a weak ϵ-neighborhood of the smooth curve $Y = \bar{Y}(x)$. To complete the proof we need only find an admissible smooth curve $Y = \mathring{Y}(x)$ for which at least

$$J[\mathring{Y}] \leq J[\bar{Y}] - \tfrac{1}{2}h$$

in the same neighborhood.

Suppose that the curve $Y = \tilde{Y}(x)$ has one corner point, $x = c$. We take the components $\tilde{y}_i(x)$ of the vector function $\tilde{Y}(x)$ and construct curves in each xz_i-plane, Figure 2,

$$\bar{D}_i: \qquad z_i = \frac{d\bar{y}_i}{dx}$$

$$\tilde{D}_i: \qquad z_i = \frac{d\tilde{y}_i}{dx}.$$

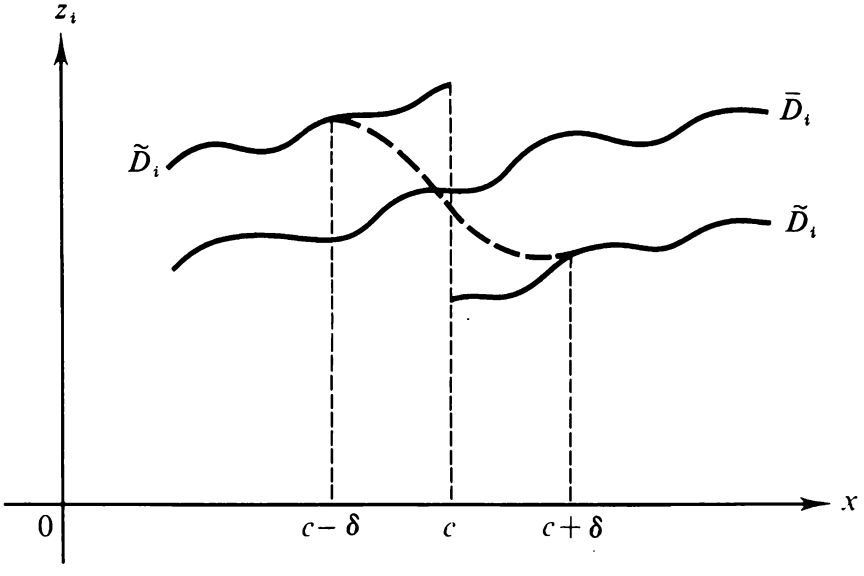

Figure 2

The curve $\tilde{D}_i$ lies in a strip of width 2ϵ along the curve $\bar{D}_i$. We take two points on the curve $\tilde{D}_i$ with abscissas $c - \delta$ and $c + \delta$, where $\delta > 0$ is sufficiently small, and connect them by constructing a curve

$$z_i = \hat{z}_i(x) \qquad (c - \delta \leq x \leq c + \delta)$$

in this strip of width 2ϵ so that

$$\int_{c-\delta}^{c+\delta} \hat{z}_i(x)\, dx = \int_{c-\delta}^{c+\delta} \frac{d\tilde{y}_i}{dx}\, dx.$$

Outside the interval $[c - \delta, c + \delta]$ we set

$$\mathring{z}_i(x) = \frac{d\tilde{y}_i}{dx}$$

and construct the function

$$\mathring{y}_i(x) = \bar{y}_i(a) + \int_a^x \mathring{z}_i(x)\, dx.$$

If δ is sufficiently small, the curve $Y = \mathring{Y}(x)$ constructed in this way lies in a weak ϵ-neighborhood of the curve $Y = \bar{Y}(x)$. It follows that

$$|J[\tilde{Y}] - J[\mathring{Y}]| \leq \int_{c-\delta}^{c+\delta} |f(x, \tilde{Y}, \tilde{Y}') - f(x, \mathring{Y}, \mathring{Y}')|\, dx \leq \tfrac{1}{2}h$$

which proves the proposition. Transition from the curve $y = \tilde{y}_i(x)$ to the curve $y = \mathring{y}_i(x)$ was accomplished by rounding off the corner.

3. THE FIRST NECESSARY CONDITION FOR EXTREMA

The classical method for solving problems of the calculus of variations resembles the approach used in differential calculus for solving elementary extremum problems. First some equations are derived (here they are differential equations); the desired function, if it exists, must satisfy these. It has been found that to furnish an extremum, a solution of these equations must also satisfy other conditions. The preliminary differential equations thus constitute the first necessary condition for an extremum.

Any necessary condition for a weak relative extremum is also a necessary condition for a strong relative extremum and *a fortiori* an absolute extremum. Hence we shall examine a weak relative extremum in order to derive the first necessary condition.

Suppose that a piecewise-smooth arc $Y = \bar{Y}(x)$, $(a \leq x \leq b)$ lies entirely in G and gives the functional $J[Y]$ a weak relative extremum. We choose some piecewise-smooth vector function*

*In this case, see equations (6a) and (6b). If $\bar{Y}(x)$ is a smooth vector function, then $H(x)$ must also be a smooth vector function.

$H(x) = \{\eta_1(x), \eta_2(x), \cdots, \eta_n(x)\}$ that satisfies the conditions

$$\eta_i(a) = \eta_i(b) = 0 \qquad (i = 1, 2, \cdots, n). \tag{1}$$

If $|\epsilon|$ is sufficiently small, the functions

$$Y = \bar{Y}(x) + \epsilon H(x) \tag{2}$$

lie in a weak neighborhood of the function $\bar{Y}(x)$. Equation (1) provides that the end-point conditions are also satisfied. Hence the functional $J[Y]$, considered over the collection of functions defined by equation (2), has an extremum for Y when $\epsilon = 0$. But

$$J[\bar{Y} + \epsilon H] = \phi(\epsilon).$$

Consequently

$$\phi'(0) = 0. \tag{3}$$

Differentiating with respect to the parameter ϵ gives the equation

$$\phi'(\epsilon) = \int_a^b \sum_{i=1}^n \{f_{y_i}(x, Y, Y')\eta_i(x) + f_{y_i'}(x, Y, Y')\eta_i'(x)\}\, dx.$$

Setting

$$f_{y_i}(x, \bar{Y}, \bar{Y}') = \bar{f}_{y_i} \qquad \text{and} \qquad f_{y_i'}(x, \bar{Y}, \bar{Y}') = \bar{f}_{y_i'},$$

we obtain

$$\phi'(0) = \int_a^b \sum_{i=1}^n \{\bar{f}_{y_i}\eta_i(x) + \bar{f}_{y_i'}\eta_i'(x)\}\, dx.$$

Thus condition (3) becomes

$$\int_a^b \sum_{i=1}^n \{\bar{f}_{y_i}\eta_i(x) + \bar{f}_{y_i'}\eta_i'(x)\}\, dx = 0. \tag{4}$$

With integration by parts, this becomes

$$\int_a^b \bar{f}_{y_i}\eta_i(x)\, dx = \left[\eta_i(x)\int_a^x \bar{f}_{y_i}\, dx\right]_{x=a}^{x=b} - \int_a^b \eta_i'(x)\left[\int_a^x \bar{f}_{y_i}\, dx\right] dx.$$

Taking equation (1) into account, we may write equation (4) in the form

$$\int_a^b \sum_{i=1}^n \left\{\bar{f}_{y_i'} - \int_a^x \bar{f}_{y_i}\, dx\right\} \eta_i'(x)\, dx = 0. \tag{5}$$

We now define the vector function $H(x)$. We set

$$\bar{f}_{y_i'} - \int_a^x \bar{f}_{y_i}\, dx = M_i(x)$$

$$c_i = \frac{1}{b-a}\int_a^b M_i(x)\, dx \tag{6a}$$

and choose

$$\eta_i(x) = \int_a^x \{M_i(t) - c_i\}\, dt. \tag{6b}$$

These are continuous functions with piecewise-continuous first derivatives. It is obvious that $\eta_i(a) = 0$ and the equations $\eta_i(b) = 0$ follow from the choice of the constants c_i. Now we write relation (5) in the form

$$\int_a^b \sum_{i=1}^{n} M_i(x)\{M_i(x) - c_i\}\, dx = 0.$$

This is equivalent to the relation

$$\int_a^b \sum_{i=1}^{n} \{M_i(x) - c_i\}^2\, dx = 0 \tag{7}$$

because the constants c_i were chosen so that

$$\int_a^b \{M_i(x) - c_i\}\, dx = 0.$$

It follows from equation (7) that

$$M_i(x) = c_i \qquad (i = 1, 2, \cdots, n)$$

which completes the derivation of the *Euler equations*:*

$$f_{y_i'} - \int_a^x f_{y_i}\, dx = c_i \qquad (i = 1, 2, \cdots, n). \tag{8}$$

*The method used to derive the Euler equations from equation (5), based on the special choice of the arbitrary functions $\eta_i(x)$, is characteristic of the calculus of variations; it is often used in this proposition. This and other fundamental propositions are called *lemmas of the calculus of variations*. Generalizations of them are presented with the necessary proofs in Sections 5, 22, 23, and 24 of the Appendix.

These equations are called the *Euler equations in integrated form.*

We can conclude that *if a piecewise-smooth curve* $Y = \bar{Y}(x)$ *lies in the region* G *and provides a weak relative extremum for the functional* $J[Y]$, *then constants* c_i *exist for which the vector function* $\bar{Y}(x)$ *satisfies the Euler equations.*

This proposition holds also when only the interior points of the curve $Y = \bar{Y}(x)$ lie in G. Note that the existence of the derivative f_x was not used in deriving the Euler equations, though it appears in the following consequence of these equations.

The vector function $\bar{Y}(x)$ *also satisfies the equation*

$$f - \sum_{i=1}^{n} y_i' f_{y_i'} - \int_a^x f_x \, dx = C \tag{9}$$

in which C *is some constant.*

PROOF: If we transform from coordinates $x, y_1, y_2, \cdots, y_n$ to other rectilinear coordinates $u, v_1, v_2, \cdots, v_n$ and the new axes make a sufficiently small angle with the original axes, the equation

$$Y = \bar{Y}(x) \qquad (a \leq x \leq b) \tag{10}$$

becomes

$$V = \bar{V}(u) \qquad (c \leq u \leq d). \tag{11}$$

Every curve $V = V(u)$ in a sufficiently small neighborhood of the curve (11) is the transform of a curve $Y = Y(x)$ that lies in a weak neighborhood of the curve (10). Having transformed the functional $J[Y]$ to new variables, we conclude that $\bar{V}(u)$ must satisfy the Euler equations in integral form for the transformed functional. By expressing these equations in the old coordinates we get equations that must be satisfied by the vector function $\bar{Y}(x)$.

We shall use the transformation of coordinates:

$$x = u + \alpha v_1, \qquad y_i = v_i \qquad (i = 1, 2, \cdots, n)$$

in which α is a constant sufficiently small in absolute value. The functional

$$\int_a^b f(x, Y, Y') \, dx$$

is transformed to

$$\int_c^d f\left(u + \alpha v_1, V, \frac{\dot{V}}{1 + \alpha \dot{v}_1}\right)(1 + \alpha \dot{v}_1)\, du$$

in which differentiation with respect to u is designated by a dot. Now we write the first Euler equation:

$$\alpha f + f_{y_1'} - \frac{\alpha}{1 + \alpha \dot{v}_1} \sum_{i=1}^{n} \dot{v}_i f_{y_i'} - \int_c^u (\alpha f_x + f_{y_1})(1 + \alpha \dot{v}_1)\, du = \text{const.}$$

The transform of this equation in the old coordinates is

$$\alpha f + f_{y_1'} - \alpha \sum_{i=1}^{n} y_i' f_{y_i'} - \int_a^x (\alpha f_x + f_{y_1})\, dx = \text{const.}$$

The vector function $\bar{Y}(x)$ must satisfy this condition. Finally, if we combine this equation with the Euler equation for $i = 1$, we can conclude that $\bar{Y}(x)$ satisfies equation (9).

4. COROLLARIES OF THE FIRST NECESSARY CONDITION

A. *Consider a piecewise-smooth vector function $\bar{Y}(x)$ that gives a weak relative extremum to the functional*

$$J[Y] = \int_a^b f(x, Y, Y')\, dx.$$

In any part of the interval $[a, b]$ in which the derivative $\bar{Y}'(x)$ is continuous, the vector function $\bar{Y}(x)$ must satisfy the Euler equations in differentiated form:

$$\frac{d}{dx} f_{y_i'} - f_{y_i} = 0 \qquad (i = 1, 2, \cdots, n) \tag{1}$$

$$\frac{d}{dx}\left(f - \sum_{i=1}^{n} y_i' f_{y_i'}\right) - f_x = 0. \tag{2}$$

Also, the Weierstrass-Erdmann corner conditions

$$\bar{f}_{y_i'}|_{x_0-0} = \bar{f}_{y_i'}|_{x_0+0} \qquad (i = 1, 2, \cdots, n) \tag{3a}$$

$$\bar{f} - \sum_{i=1}^{n} \bar{y}_i' \bar{f}_{y_i'}|_{x_0-0} = \bar{f} - \sum_{i=1}^{n} \bar{y}_i' \bar{f}_{y_i'}|_{x_0+0} \tag{3b}$$

hold at any corner point (a point of discontinuity of $\bar{Y}'$) x_0 *of* $\bar{Y}(x)$.

PROOF: If $\bar{Y}'(x)$ is continuous in an interval, then the functions $\bar{f}_{y_i}$ and $\bar{f}_x$ are continuous in the same interval. Hence the functions

$$\int_a^x \bar{f}_{y_i}\,dx \qquad \text{and} \qquad \int_a^x \bar{f}_x\,dx \tag{4}$$

have continuous derivatives of the first order in this interval. Consequently we can differentiate the Euler equations and equation (9), Section 3, after replacing Y by $\bar{Y}(x)$. The Weierstrass-Erdmann corner conditions (3a) and (3b) follow from the fact that the quantities (4) are continuous at every corner point; thus the quantities

$$\bar{f}_{y_i'} \qquad \text{and} \qquad \bar{f} - \sum_{i=1}^{n} \bar{y}_i' \bar{f}_{y_i'}$$

[equal to the expressions (3a) and (3b), respectively] are also continuous.

Every smooth solution of the Euler equations [which imply equation (1)] is called an *extremal* of the functional $J[Y]$. Every curve composed of parts of extremals and satisfying the Weierstrass-Erdmann corner conditions is called a *broken extremal.**

B. *If the integrand function* $f(x, Y, Y')$ *does not contain the components* y_i *of the vector* Y, *but does contain* y_i', *the Euler equations have a first integral:*

$$f_{y_i'} = c_i.$$

If the functional has the form

$$\int_a^b f(Y, Y')\,dx,$$

then a vector function $\bar{Y}(x)$ *that supplies it with a weak relative extremum satisfies the equation*

$$f - \sum_{i=1}^{n} y_i' f_{y_i'} = C.$$

These deductions have special significance for the case $n = 1$ in which the extremal is defined by one equation.

*Extremals that are not broken are called *smooth extremals.*

C. *In the class of twice-differentiable functions $Y(x)$, equations* (1) *and* (2) *are differential equations of the second order, and equation* (2) *is a corollary of equation* (1).

PROOF: A proof is needed only for the second assertion. If we carry out the differentiation in equation (2), we have

$$\sum_{i=1}^{n} y_i' \left(f_{y_i} - \frac{d}{dx} f_{y_i'} \right) = 0. \tag{5}$$

This shows that equation (2) is a consequence of equation (1). Equation (5) allows the following inference: *Suppose a twice differentiable vector $Y(x)$ satisfies equation* (2) *in some interval and also satisfies all but the kth of the equations*

$$\frac{d}{dx} f_{y_i'} - f_{y_i} = 0 \qquad (i = 1, 2, \cdots, n).$$

If $y_k'(x) \neq 0$ in the interval then $Y(x)$ satisfies the kth equation as well.

5. INVARIANCE OF THE EULER EQUATION

Replace the independent variable x by the expression

$$x = \omega(u) \tag{1}$$

and suppose $\omega(u)$ has a continuous derivative that is not zero; assume that it is positive in some interval $c \leq u \leq d$ which equation (1) transforms to $a \leq x \leq b$. By means of the relations

$$V = V(u) = Y(\omega(u)) = Y(x) = Y$$

the functional

$$J[Y] = \int_a^b f(x, Y, Y')\, dx \tag{2}$$

becomes the functional

$$\int_c^d g(u, V, V')\, du \tag{3}$$

in which

$$g(u, V, W) = f\left(\omega(u), V, \frac{W}{\omega'(u)}\right)\omega'(u).$$

The boundary conditions

$$Y(a) = A \qquad \text{and} \qquad Y(b) = B \tag{4}$$

become

$$V(c) = A \qquad \text{and} \qquad V(d) = B. \tag{5}$$

The integrand function $g(u, V, W)$ may not have a derivative with respect to u; to give it a certain number of derivatives with respect to this variable, we must assume that the function $\omega(u)$ has the same number of derivatives. Here we assume that the integrand function of the original functional is differentiable a sufficient number of times.

Suppose the vector function $\bar{Y}(x)$ satisfies the Euler equations* for the functional (2), and also the boundary conditions (4). We introduce the vector function $\bar{V}(u) = \bar{Y}(\omega(u))$ which satisfies condition (5). If the vector function $\bar{V}(u)$ satisfies the Euler equations for the functional (3), we can assert that the ability to satisfy the Euler equations is an *invariant property* of the vector function under the given transformation.

To demonstrate that this is an invariant property, we introduce an arbitrary piecewise-smooth vector function $Z(u) = \{\zeta_1(u), \zeta_2(u), \cdots, \zeta_n(u)\}$, $(c \leq u \leq d)$ equal to zero at the ends of the interval. Using equation (1), we express u in terms of x so that $Z(u) = H(x)$. Here we introduce the function

$$\phi(\epsilon) = \int_c^d g\left(u, \bar{V} + \epsilon Z, \frac{d\bar{V}}{du} + \epsilon\frac{dZ}{du}\right)du$$

which can also be expressed in the form

$$\phi(\epsilon) = \int_a^b f\left(x, \bar{Y} + \epsilon H, \frac{d\bar{Y}}{dx} + \epsilon\frac{dH}{dx}\right)dx.$$

Since $\bar{Y}(x)$ satisfies the Euler equations for the functional (2), we

*In general, the equations in integrated form should be used if the function $\bar{Y}(x)$ is not smooth.

conclude from the second expression for $\phi(\epsilon)$ that $\phi'(0) = 0$. Hence

$$\int_c^d \sum_{i=1}^n \{\bar{g}_{v_i}\zeta_i(u) + \bar{g}_{v_i'}\zeta_i'(u)\}\, du = 0.$$

Thus we can show, by the method used in Section 3, that the vector function $\bar{V}(u)$ satisfies the Euler equations in integral form for the functional (3).

Under certain conditions, the Euler equations are invariant under more general transformations. Both the vector function Y and the independent variable x appear in the equations of these transformations. Consider the transformation given by the equations

$$\begin{aligned} x &= \phi(u, v_1, v_2, \cdots, v_n) \equiv \phi(u, V), \\ y_i &= \psi_i(u, v_1, v_2, \cdots, v_n) \equiv \psi_i(u, V) \qquad (6) \\ &\qquad (i = 1, 2, \cdots, n). \end{aligned}$$

Assume that these formulas define a one-to-one mapping that is thrice continuously differentiable; it maps some region H of the space of variables $u, v_1, v_2, \cdots, v_n$ onto a region $G' \subset G$. Moreover, suppose that in the region H

$$\frac{\partial(\phi, \psi_1, \psi_2, \cdots, \psi_n)}{\partial(u, v_1, v_2, \cdots, v_n)} \not\equiv 0. \qquad (7)$$

Consider the functional (2) with boundary conditions (4) and suppose that the points (a, A) and (b, B) lie in the region G'. The transformation (6) maps (a, A) and (b, B) into the points $(c, c_1, c_2, \cdots, c_n)$, that is, (c, C) and $(d, d_1, d_2, \cdots, d_n)$, that is, (d, D), respectively; without loss of generality we may assume that $c < d$. If we perform the transformation (6), the functional (2) becomes a line integral. We shall consider this line integral only for the curves $V = V(u)$, $(c \leq u \leq d)$ lying in the region H; it is a new functional:

$$J_1[V] = \int_c^d F\left(u, V, \frac{dV}{du}\right) du.$$

The functions $\phi(u, V)$ and $\psi_i(u, V)$ have been assumed to be

thrice continuously differentiable; thus $F(u, V, W)$ has continuous partial derivatives of the second order with respect to each of its arguments.*

Suppose the curve $Y = \bar{Y}(x), \quad (a \leq x \leq b)$ satisfies the Euler equations and the boundary conditions, that it lies in the region G', and that its image is the curve $V = \bar{V}(u)$. We shall show that the vector function $\bar{V}(u)$ satisfies the Euler equations for the new functional $J_1[V]$. For this proof we must assume that the vector function $\bar{V}(u)$ is twice continuously differentiable; thus $\bar{Y}(x)$ also has this property.

We introduce a vector function $Z(u) = \{\zeta_1(u), \zeta_2(u), \cdots, \zeta_n(u)\}$ which is twice differentiable in the interval $[c, d]$ and equal to zero at the ends of the interval. We then write:

$$V = \bar{V}(u) + \epsilon Z(u). \tag{8}$$

The transformation (6) transforms this curve to a curve with the parametric equations:

$$x = \phi(u, \bar{V}(u) + \epsilon Z(u)) \qquad (c \leq u \leq d) \tag{9a}$$

$$y_i = \psi_i(u, \bar{V}(u) + \epsilon Z(u)) \qquad (i = 1, 2, \cdots, n). \tag{9b}$$

For $\epsilon = 0$, these can be replaced by ordinary equations of the form:

$$y_i = \bar{y}_i(x) \qquad (a \leq x \leq b) \qquad (i = 1, 2, \cdots, n).$$

Consequently

$$y_i'(x) \frac{d}{du} \phi(u, \bar{V}(u)) = \frac{d}{du} \psi_i(u, \bar{V}(u)) \qquad (i = 1, 2, \cdots, n).$$

We conclude that the quantity

$$\frac{d}{du} \phi(u, \bar{V}(u)) \tag{10}$$

vanishes at no point of the interval $[c, d]$. If this quantity vanished

*These second partial derivatives are used to derive the Euler equations in differentiated form. The remainder of the proof can be made with $\phi(u, V)$ and $\psi_i(u, V)$ twice continuously differentiable.

at any point in $[c, d]$, the equations:

$$\frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v_1}\bar{v}_1' + \frac{\partial \phi}{\partial v_2}\bar{v}_2' + \cdots + \frac{\partial \phi}{\partial v_n}\bar{v}_n' = 0,$$

$$\frac{\partial \psi_i}{\partial u} + \frac{\partial \psi_i}{\partial v_1}\bar{v}_1' + \frac{\partial \psi_i}{\partial v_2}\bar{v}_2' + \cdots + \frac{\partial \psi_i}{\partial v_n}\bar{v}_n' = 0$$

$$(i = 1, 2, \cdots, n).$$

would hold at that point. This contradicts equation (7).

We now use a theorem on implicit functions; since the quantity (10) is different from zero everywhere in the interval $[c, d]$, equation (9a) determines a function $u = u(x, \epsilon)$; this function has continuous second derivatives over the interval $a \leq x \leq b$ for $|\epsilon|$ sufficiently small. Thus, for sufficiently small values of $|\epsilon|$, the transformation (9a) and (9b) can be expressed in ordinary form:

$$Y = \Omega(x, \epsilon).$$

Here the vector function $\Omega(x, \epsilon) = \{\omega_1(x, \epsilon), \omega_2(x, \epsilon), \cdots, \omega_n(x, \epsilon)\}$ has continuous derivatives of the second order and satisfies the equations

$$\Omega(a, \epsilon) = A \qquad \text{and} \qquad \Omega(b, \epsilon) = B.$$

The curves $Y = \Omega(x, \epsilon)$ are images of the curves given by equation (8); they lie in a weak neighborhood of the curve $Y = \bar{Y}(x)$ if $|\epsilon|$ is sufficiently small. We now introduce the function

$$\Phi(\epsilon) = \int_a^b f(x, \Omega(x, \epsilon), \Omega_x(x, \epsilon))\, dx$$

which can also be written

$$\Phi(\epsilon) = \int_c^d F(u, \bar{V}(u) + \epsilon Z(u), \bar{V}'(u) + \epsilon Z'(u))\, du.$$

From the first of these expressions

$$\Phi'(0) = \int_a^b \sum_{i=1}^{n} \{\bar{f}_{y_i}\omega_{i\epsilon}(x, 0) + \bar{f}_{y_i'}\omega_{ix\epsilon}(x, 0)\}\, dx$$

and from the second

$$\Phi'(0) = \int_c^d \sum_{i=1}^n \{\bar{F}_{v_i}\zeta_i(u) + \bar{F}_{v_i'}\zeta_i'(u)\}\, du.$$

Since for arbitrary ϵ

$$\omega_i(a, \epsilon) = a_i \qquad \text{and} \qquad \omega_i(b, \epsilon) = b_i \qquad (i = 1, 2, \cdots, n),$$

it follows that

$$\omega_{i\epsilon}(a, \epsilon) = 0 \qquad \text{and} \qquad \omega_{i\epsilon}(b, \epsilon) = 0 \qquad (i = 1, 2, \cdots, n).$$

Hence, after integration by parts, the first expression for $\Phi'(0)$ becomes

$$\Phi'(0) = \int_a^b \sum_{i=1}^n \omega_{i\epsilon}(x, 0) \left\{\bar{f}_{y_i} - \frac{d}{dx}\bar{f}_{y_i'}\right\} dx.$$

The function $\bar{Y}(x)$ satisfies the Euler equations for the original functional; thus $\Phi'(0) = 0$. The second expression implies that

$$\int_c^d \sum_{i=1}^n \{\bar{F}_{v_i}\zeta_i(u) + \bar{F}_{v_i'}\zeta_i'(u)\}\, du = 0. \tag{11}$$

Here $Z(u)$ is an arbitrary vector function that has a continuous second derivative; at the end points of the interval, the function vanishes. Integration by parts (as in Section 3) yields

$$\bar{F}_{v_i'} - \int_c^u \bar{F}_{v_i}\, du = K_i \qquad (i = 1, 2, \cdots, n)$$

in which K_i is a constant. It follows that

$$\frac{d}{du}\bar{F}_{v_i'} - \bar{F}_{v_i} = 0 \qquad (i = 1, 2, \cdots, n).$$

This shows that $\bar{V}(u)$ satisfies the Euler equations for the transformed functional.

Notice that it is not necessary to proceed as in Section 3 after deriving equation (11). Instead, since $\bar{V}(u)$ has a continuous second derivative, we can integrate by parts in another way to obtain

$$\int_c^d \sum_{i=1}^n \zeta_i(u) \left\{\bar{F}_{v_i} - \frac{d}{du}\bar{F}_{v_i'}\right\} du = 0.$$

Here $\zeta_i(u)$, $(i = 1, 2, \cdots, n)$ is an arbitrary function that has a continuous second-order derivative; the function equals zero at the end points of the interval $[c, d]$. To show that the relation

$$\bar{F}_{v_i} = \frac{d}{du}\bar{F}_{v_i'} \qquad (i = 1, 2, \cdots, n)$$

follows from this equation we must prove the proposition known as the *lemma of Lagrange*. This is usually called the first or fundamental lemma of the calculus of variations.

LEMMA OF LAGRANGE: *If $M(x)$ is a continuous function on the interval $[\alpha, \beta]$ and if the equation*

$$\int_\alpha^\beta M(x)\eta(x)\,dx = 0$$

holds for an arbitrary function $\eta(x)$ that has continuous derivatives of the rth order and that vanishes together with all its derivatives up to and including those of order k, $k \leq r$, at the end points of $[\alpha, \beta]$, then $M(x) \equiv 0$.

To prove this lemma we suppose that $M(x) \not\equiv 0$ at some point of the interval $[\alpha, \beta]$. The continuity of $M(x)$ requires the existence of some interval $[x_1, x_2] \subset [\alpha, \beta]$ in which $M(x)$ is never zero. We set

$$\tilde{\eta}(x) = 0 \qquad \text{outside the interval } [x_1, x_2]$$

and

$$\tilde{\eta}(x) = [(x_2 - x)(x - x_1)]^{r+1} \quad \text{for} \quad x_1 \leq x \leq x_2.$$

Obviously $\tilde{\eta}(x)$ is continuously differentiable r times in the interval $[\alpha, \beta]$ and vanishes with all its derivatives up to and including the kth (even up to the rth) at the end points of the interval $[\alpha, \beta]$. Hence

$$0 = \int_\alpha^\beta M(x)\tilde{\eta}(x)\,dx = \int_{x_1}^{x_2} M(x)[(x_2 - x)(x - x_1)]^{r+1}\,dx \neq 0.$$

This contradiction proves the lemma.

6. A THEOREM OF HILBERT

Hilbert proposed that *if* $Y = \bar{Y}(x)$ *is an extremal for the functional*

$$J[Y] = \int_a^b f(x, Y, Y')\, dx$$

and if at some point $x = c \in (a, b)$

$$|f_{y_i' y_j'}|_{i,j=1}^{n} \neq 0 \tag{1}$$

then, in some neighborhood of the point $x = c$, *the vector function* $\bar{Y}(x)$ *has a continuous second derivative.*

PROOF: On the basis of the integral equation

$$f_{y_i'}(x, \bar{Y}(x), \bar{Y}'(x)) = \int_a^x f_{y_i}(t, \bar{Y}(t), \bar{Y}'(t))\, dt + c_i \qquad (i = 1, 2, \cdots, n) \tag{2}$$

the system

$$f_{y_i'}(x, \bar{Y}(x), Z) = \int_a^x f_{y_i}(t, \bar{Y}(t), \bar{Y}'(t))\, dt + c_i \qquad (i = 1, 2, \cdots, n)$$

can be solved for $z_1, z_2, \cdots, z_n$, for every x. Condition (1) expresses the fact that at $x = c$ the Jacobian of this system is different from zero. According to a theorem on implicit functions, this system and the conditions

$$z_1(c) = \bar{y}_1'(c), \qquad z_2(c) = \bar{y}_2'(c), \cdots, z_n(c) = \bar{y}_n'(c)$$

define the functions $z_1(x), z_2(x), \cdots, z_n(x)$ in some neighborhood of the point $x = c$ and provide that they are continuously differentiable in this neighborhood. Thus $z_i(x) = \bar{y}_i'(x)$, $(i = 1, 2, \cdots, n)$ and the derivatives $\bar{y}_i''(x)$, $(i = 1, 2, \cdots, n)$ exist and are continuous in some neighborhood of the point $x = c$. This proves Hilbert's theorem.

Assuming that condition (1) holds at the point $x = c$, we can carry out the differentiation of equation (2) in the neighborhood

of this point. Differentiation yields

$$\bar{f}_{y_i'x} + \sum_{j=1}^{n} \bar{f}_{y_i'y_j}\bar{y}_j'(x) + \sum_{j=1}^{n} \bar{f}_{y_i'y_j'}\bar{y}_j''(x) = \bar{f}_{y_i} \qquad (i = 1, 2, \cdots, n).$$

Upon setting

$$D(x, Y, Z) = |f_{y_i'y_j'}(x, Y, Z)|_{i,j=1}^{n},$$

we find that in some neighborhood of the point $x = c$, the following equation holds:

$$\bar{y}_i''(x) = \sum_{j=1}^{n} \frac{A_{ij}(x, \bar{Y}, \bar{Y}')}{D(x, \bar{Y}, \bar{Y}')}\bar{y}_j'(x) + \sum_{j=1}^{n} \frac{B_{ij}(x, \bar{Y}, \bar{Y}')}{D(x, \bar{Y}, \bar{Y}')}\{\bar{f}_{y_j} - \bar{f}_{y_j'x}\} \qquad (i = 1, 2, \cdots, n).$$

The quantities

$$A_{ij}(x, Y, Z) \qquad \text{and} \qquad B_{ij}(x, Y, Z)$$

are polynomials whose arguments are second-order partial derivatives of $f(x, Y, Z)$.

Repeatedly in what follows we shall wish to assume that the determinant $D(x, Y, Z)$ is not zero for some values of x, Y, Z; for this reason we introduce the following terminology.

A system of $2n + 1$ numbers

$$(x_0, y_{01}, y_{02}, \cdots, y_{0n}, z_{01}, z_{02}, \cdots, z_{0n}) = (x_0, Y_0, Z_0) \quad (3)$$

in which (x_0, Y_0) is a point of the region G and $z_{01}, z_{02}, \cdots, z_{0n}$ are finite is called an *element.*

If $D(x_0, Y_0, Z_0) \neq 0$, the element (x_0, Y_0, Z_0) is called *nonsingular* with reference to the given functional. If every element $(x, \bar{Y}(x), \bar{Y}'(x))$ of some extremal arc is nonsingular we call it a nonsingular extremal. Finally, if every element (x, Y, Z) is nonsingular, the functional is called nonsingular.

7. THE EXISTENCE OF EXTREMALS

THEOREM: *Consider*

$$(x_0, Y_0, Y_0') = (x_0, y_{01}, y_{02}, \cdots, y_{0n}, y_{01}', y_{02}', \cdots, y_{0n}') \quad (1)$$

a nonsingular element for the functional

$$J[Y] = \int_a^b f(x, Y, Y')\, dx.$$

In this case there exists only one smooth vector function $Y = Y(x)$ *that is defined in some neighborhood of the point* $x = x_0$, *satisfies the equation*

$$\frac{d}{dx} f_{y_i'} - f_{y_i} = 0 \qquad (i = 1, 2, \cdots, n), \tag{2}$$

and contains the element (1). *In other words, only one smooth vector function that satisfies equation* (2) *has the property that*

$$Y(x_0) = Y_0 \qquad \textit{and} \qquad Y'(x_0) = Y_0'. \tag{3}$$

PROOF: Let us recall the quantities $A_{ij}(x, Y, Y')$ and $B_{ij}(x, Y, Y')$ introduced in Section 6; then consider the system of differential equations:

$$\begin{aligned} y_i'' = \frac{1}{D(x, Y, Y')} \Bigg\{ & \sum_{j=1}^{n} A_{ij}(x, Y, Y') y_j' \\ & + \sum_{j=1}^{n} B_{ij}(x, Y, Y')(f_{y_j} - f_{y_j'x}) \Bigg\} \\ \equiv F_i(x, Y, Y') \qquad & (i = 1, 2, \cdots, n). \end{aligned} \tag{4}$$

For $n = 1$ this reduces to a single equation:

$$\begin{aligned} y'' &= F(x, y, y') \\ &\equiv \frac{f_y(x, y, y') - f_{y'x}(x, y, y') - y' f_{y'y}(x, y, y')}{f_{y'y'}(x, y, y')}. \end{aligned} \tag{4'}$$

In some neighborhood of the element (1), the right side of equation (4) is continuous; in this neighborhood it satisfies a Cauchy-Lipschitz condition with respect to the arguments Y, Y'. Existence theorems for ordinary differential equations provide the system (4) with only one solution that satisfies condition (3) in some neighborhood of x_0; this solution $Y = Y(x)$ also satisfies equation (2). In this way we have shown that the system (2) has a smooth solution that satisfies condition (3); it remains for us to prove that this solution is unique.

Hilbert's theorem is applicable here because the element (1) is nonsingular; it states that every smooth solution of the system (2) that satisfies conditions (3) has a second derivative in some neighborhood of the point x_0 and therefore satisfies the system (4) which has only one solution. This completes our proof.

By transforming the Euler system (2) to the system (4) we change it to a *normal system* of $2n$ equations of the first order:

$$y_i' = z_i, \qquad z_i' = F_i(x, Y, Z) \qquad (i = 1, 2, \cdots, n).$$

We now use another method to transform the Euler equations to a normal system of first-order equations. This is known as a transformation to *canonical form*. The symbol Δ denotes the neighborhood of the element (1) consisting exclusively of nonsingular elements (x, Y, Z) of our functional. We consider the vector function $V = \{v_1, v_2, \cdots, v_n\}$ in Δ with components

$$v_i = f_{y_i'}(x, Y, Z) \qquad (i = 1, 2, \cdots, n). \tag{5a}$$

The formulas

$$x = x, \qquad y_i = y_i, \qquad v_i = f_{y_i'}(x, Y, Z) \qquad (i = 1, 2, \cdots, n) \tag{5b}$$

transform Δ into a region Γ of the space $x, y_1, y_2, \cdots, y_n, v_1, v_2, \cdots, v_n$. Because the element (1) is nonsingular, we may assume that the mapping (5b) is one-to-one and, if necessary, decrease the neighborhood Δ. This allows us to consider Z as a vector function in the region Γ:

$$z_i = p_i(x, Y, V) \qquad (i = 1, 2, \cdots, n). \tag{6}$$

Note that the functions $p_i(x, Y, V)$ are twice continuously differentiable because the right-hand side of formula (5a) has this property; also, the Jacobian of this right member relative to $z_1, z_2, \cdots, z_n$ is different from zero everywhere in the region Δ.

Replacing the vector Z by means of equation (6) we form the function $H(x, Y, V)$:

$$H = f(x, Y, Z) - \sum_{j=1}^{n} z_j f_{y_j'}(x, Y, Z) = f(x, Y, Z) - \sum_{j=1}^{n} z_j v_j. \tag{7}$$

If we differentiate equation (7) with respect to the variables

x, y_i, v_i we find that

$$\frac{\partial H}{\partial x} = f_x + \sum_{j=1}^{n} f_{y_j'} \frac{\partial p_j}{\partial x} - \sum_{j=1}^{n} \frac{\partial p_j}{\partial x} v_j = f_x \tag{8a}$$

$$\frac{\partial H}{\partial y_i} = f_{y_i} + \sum_{j=1}^{n} f_{y_j'} \frac{\partial p_j}{\partial y_i} - \sum_{j=1}^{n} \frac{\partial p_j}{\partial y_i} v_j = f_{y_i} \tag{8b}$$

$$\frac{\partial H}{\partial v_i} = \sum_{j=1}^{n} f_{y_j'} \frac{\partial p_j}{\partial v_i} - \sum_{j=1}^{n} \frac{\partial p_j}{\partial v_i} v_j - p_i = -p_i. \tag{8c}$$

Now we transform the Euler system to the variables x, Y, V. From equations (5a) and (8b), the equation

$$\frac{d}{dx} f_{y_i'} - f_{y_i} = 0 \qquad (i = 1, 2, \cdots, n)$$

becomes

$$\frac{dv_i}{dx} = \frac{\partial H}{\partial y_i} \qquad (i = 1, 2, \cdots, n)$$

with the condition that $y_i' = z_i$, $(i = 1, 2, \cdots, n)$. On the basis of equations (8c) and (6) this condition can be written in the form

$$\frac{dy_i}{dx} = -\frac{\partial H}{\partial v_i} \qquad (i = 1, 2, \cdots, n).$$

The Euler system of equations can thus be expressed in *canonical form:*

$$\frac{dv_i}{dx} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dx} = -\frac{\partial H}{\partial v_i} \qquad (i = 1, 2, \cdots, n).$$

We have the *canonical variables* Y and V instead of the variables Y and $Z = Y'$.

The theorem we have established of the existence of extremals has a *local* character. The question of the existence of a *global* extremal joining two arbitrary points of a domain presents considerable difficulty; it has a solution only with some additional assumptions about the function $f(x, Y, Z)$ and about the domain.

8. DEPENDENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS ON PARAMETERS

The propositions with which the present section is concerned belong to the theory of ordinary differential equations. They are introduced here for the convenience of the reader. One is given with a proof and the other without proof.*

THEOREM 1: *Suppose a system of equations*

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n) \tag{1}$$

has a solution

$$y_i = y_i(x) \qquad (a \leq x \leq b) \qquad (i = 1, 2, \cdots, n). \tag{2}$$

In some neighborhood D of the curve (2), the functions $f_i(x, y_1, y_2, \cdots, y_n)$ are continuous and satisfy a Cauchy-Lipschitz condition with respect to the variables y_i; that is,

$$|f_i(x, y_1, y_2, \cdots, y_n) - f_i(x, \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n)| \leq \frac{K}{\sqrt{n}} \sqrt{(y_1 - \bar{y}_1)^2 + \cdots + (y_n - \bar{y}_n)^2},$$

where K is some constant. These conditions provide the existence of a neighborhood $D' \subset D$ of the curve (2) such that only one integral curve

$$y_i = y_i(x; \xi, \eta_1, \eta_2, \cdots, \eta_n) \qquad (a \leq x \leq b) \qquad (i = 1, 2, \cdots, n)$$

of the system (1) passes through a point $(\xi, \eta_1, \eta_2, \cdots, \eta_n) \in D'$. This theorem is often called the imbedding theorem because it establishes that the integral curve is a member of some family of integral curves.

---

*A proof appears in "*Lekcii po teorii obyknovennyh differencial'nyh uravnenii*" (Lectures on the Theory of Ordinary Differential Equations) by I. G. Petrovskiĭ, Moscow and Leningrad, 1952. Proofs of existence theorems for implicit functions and differential equations and proofs of imbedding theorems can also be found in "Lectures on the Calculus of Variations" by G. A. Bliss, University of Chicago Press, Chicago, 1946.

PROOF: For simplicity we restrict the proof to the case in which $n = 1$. The function

$$y = y(x) \tag{2'}$$

which is a solution of the equation

$$\frac{dy}{dx} = f(x, y) \tag{1'}$$

evidently satisfies the integral equation

$$y(x) = y(\xi) + \int_{\xi}^{x} f(t, y(t))\, dt \qquad (a \leq x \leq b).$$

We use the method of successive approximations; starting with the zero approximation

$$y_0(x; \xi, \eta) = \eta + y(x) - y(\xi),$$

we set

$$y_{m+1}(x; \xi, \eta) = \eta + \int_{\xi}^{x} f(t, y_m(t; \xi, \eta))\, dt \qquad (m = 0, 1, 2, \cdots).$$

Here it is assumed that

$$a \leq \xi \leq b \qquad \text{and} \qquad |\eta - y(\xi)| \leq \epsilon e^{-Kl}$$

in which $l = b - a$; thus if $\epsilon > 0$ is sufficiently small, the point (ξ, η) lies sufficiently close to the curve (2').

First we show that the following inequality holds

$$|y_m(x; \xi, \eta) - y(x)| \leq \epsilon \qquad (a \leq x \leq b).$$

This implies that all curves $y = y_m(x; \xi, \eta)$, $(a \leq x \leq b)$ will lie in the region D for sufficiently small values of $\epsilon > 0$. For such a value of ϵ we determine a region $D' \subset D$. If we set

$$y_{-1}(x; \xi, \eta) = y(x),$$

we get two inequalities:

$$|y_0(x; \xi, \eta) - y(x)| = |\eta - y(\xi)| \leq \epsilon e^{-Kl}$$

and

$$|y_m(x;\xi,\eta) - y_{m-1}(x;\xi,\eta)| \leq K\left|\int_\xi^x |y_{m-1}(t;\xi,\eta) - y_{m-2}(t;\xi,\eta)|\,dt\right| \qquad (m = 1, 2, \cdots).$$

Hence

$$|y_m(x;\xi,\eta) - y_{m-1}(x;\xi,\eta)| \leq \epsilon e^{-Kl}K^m\frac{|x-\xi|^m}{m!} \leq \epsilon e^{-Kl}\frac{K^m l^m}{m!}. \tag{3}$$

This implies

$$|y_m(x;\xi,\eta) - y(x)| \leq \sum_{j=0}^{m} |y_j(x;\xi,\eta) - y_{j-1}(x;\xi,\eta)| \leq \epsilon e^{-Kl}\sum_{j=0}^{m}\frac{K^j l^j}{j!} \leq \epsilon \qquad (a \leq x \leq b)$$

which completes the proof.

Inequality (3) shows that the series

$$y_0(x;\xi,\eta) + \{y_1(x;\xi,\eta) - y_0(x;\xi,\eta)\} + \{y_2(x;\xi,\eta) - y_1(x;\xi,\eta)\} + \cdots$$

converges uniformly over the entire interval $a \leq x \leq b$. The sum of this series is $\lim_{m\to\infty} y_m(x;\xi,\eta)$, the solution $y(x;\xi,\eta)$ of equation (1′). This establishes the existence of the solution; its uniqueness can also be proved.*

THEOREM 2: *Suppose a system of equations*

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n) \tag{1}$$

has a solution

$$y_i = y_i(x) \qquad (a \leq x \leq b) \qquad (i = 1, 2, \cdots, n) \tag{2}$$

*See Bliss, Gilbert A., "Lectures on the Calculus of Variations," University of Chicago Press, Chicago, 1946, Appendix.

such that in some neighborhood D of the solution, each $f_i(x, y_1, y_2, \cdots, y_n)$ has continuous partial derivatives of order $\leq q$ in all the variables $x, y_1, y_2, \cdots, y_n$. These conditions provide a neighborhood $D'' \subset D'$ of the solution in which each of the functions of Theorem 1 $y_i(x; \xi, \eta_1, \eta_2, \cdots, \eta_n)$ and its derivatives $y'_{ix}(x; \xi, \eta_1, \eta_2, \cdots, \eta_n)$ has a continuous partial derivative of an order $\leq q$ with respect to the variables $x; \xi, \eta_1, \eta_2, \cdots, \eta_n$, where $a \leq x \leq b$ and $(\xi, \eta_1, \eta_2, \cdots, \eta_n) \in D''$.

COROLLARY: *Suppose a system of equations*

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n, \sigma_1, \sigma_2, \cdots, \sigma_j) \qquad (i = 1, 2, \cdots, n) \quad (4)$$

has a solution

$$y_i = y_i(x) \qquad (a \leq x \leq b) \qquad (i = 1, 2, \cdots, n) \qquad (5)$$

for $\sigma_1 = \sigma_2 = \cdots = \sigma_j = 0$. Assume that in some neighborhood D of this solution, for $|\sigma_k| < \alpha$, $(k = 1, 2, \cdots, j)$, each of the $f_i(x, y_1, y_2, \cdots, y_n, \sigma_1, \sigma_2, \cdots, \sigma_j)$ has continouus partial derivatives of order $\leq q$ in all the variables $x, y_1, y_2, \cdots, y_n, \sigma_1, \sigma_2, \cdots, \sigma_j$. These conditions provide a neighborhood $D^* \subset D$ of the solution and a positive number $\alpha' \leq \alpha$ such that only one integral curve

$$y_i = y_i(x; \xi, \eta_1, \eta_2, \cdots, \eta_n; \sigma_1, \sigma_2, \cdots, \sigma_j) \equiv y_i(x; \xi, \eta; \sigma)$$

passes through any point $(\xi, \eta_1, \eta_2, \cdots, \eta_n) \in D^*$ if the σ_k satisfy the inequality

$$|\sigma_k| < \alpha' \qquad (k = 1, 2, \cdots, j),$$

where the functions $y_i(x; \xi, \eta; \sigma)$ and their derivatives $y'_{ix}(x; \xi, \eta; \sigma)$ $(a \leq x \leq b)$, $(\xi, \eta_1, \eta_2, \cdots, \eta_n) \in D^*$ have continuous partial derivatives of order $\leq q$ in all the variables $x; \xi, \eta_1, \eta_2, \cdots, \eta_n; \sigma_1, \sigma_2, \cdots, \sigma_j$.

The proof follows immediately by applying the preceding theorem to the system

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n; \sigma_1, \sigma_2, \cdots, \sigma_j) \qquad (i = 1, 2, \cdots, n)$$

$$\frac{d\sigma_k}{dx} = 0 \qquad (k = 1, 2, \cdots, j).$$

9. A THEOREM OF BERNSTEIN

If $n = 1$ and the functional is nonsingular, we have one Euler equation of the form

$$y'' = F(x, y, y'). \tag{1}$$

We shall prove a theorem of S. N. Bernstein on the existence of a global solution of this equation.*

THEOREM: *Suppose the functions F, F_y, $F_{y'}$ are continuous at every finite point (x, y) for arbitrary finite values of y'. Assume that functions*

$$\alpha = \alpha(x, y) \geq 0, \qquad \beta = \beta(x, y) \geq 0$$

and a constant $k > 0$ such that

$$F_y(x, y, y') > k \tag{2}$$

$$|F(x, y, y')| \leq \alpha y'^2 + \beta \tag{3}$$

exist and α and β are bounded in every finite part of the plane. If these conditions hold then one and only one integral curve of equation (1) passes through any two points (a, a_1) and (b, b_1) having different abscissas, $(a \neq b)$. The proof of this theorem consists of three parts.

A. First we show that not more than one integral curve passes through the points (a, a_1) and (b, b_1). For this we use only condition (2); condition (3) is not necessary.

Suppose there are two integral curves $y = \phi_1(x)$ and $y = \phi_2(x)$. We set $\delta = \delta(x) = \phi_2(x) - \phi_1(x)$. Then

$$\delta'' = F(x, \phi_2, \phi_2') - F(x, \phi_1, \phi_1') = \delta F_y^* + \delta' F_{y'}^*,$$

where

$$F_y^* = F_y(x, \phi_1 + \theta\delta, \phi_1' + \theta\delta'),$$

$$F_{y'}^* = F_{y'}(x, \phi_1 + \theta\delta, \phi_1' + \theta\delta')$$

*We refer to a memoir written by S. N. Bernstein in 1912. It appeared under the title "Ob uravnenniya variacionnogo iscisleniya" (On Equations of the Calculus of Variations) in USPEHI MATEMATICESKIH NAUK Vol. 8, 1941. It discusses both systems of ordinary differential equations and equations with partial derivatives. We restrict ourselves to the simplest proposition of this paper.

and $0 < \theta < 1$. If $\phi_2(x) \not\equiv \phi_1(x)$ then the difference $\delta(x)$ which is equal to zero at the ends of the interval $[a, b]$ must have a positive maximum or a negative minimum in the interval (a, b), say at the point ξ. In the first case at the point ξ

$$\delta'' \leq 0, \qquad \delta > 0, \qquad \delta' = 0$$

which is impossible because $F_y^* > k > 0$; in the second case at the point ξ

$$\delta'' \geq 0, \qquad \delta < 0, \qquad \delta' = 0$$

which is impossible for the same reason. Uniqueness has thus been proved.

B. We now verify two evaluations: if a function $y(x)$ satisfies equation (1) over the interval $[a, c]$ and satisfies $y(a) = a_1$, $y(c) = c_1$, then over the whole interval

$$\left| y(x) - \frac{a_1(c-x) + c_1(x-a)}{c-a} \right| \leq \frac{1}{k} \max_{a \leq x \leq c} \left| F\left(x, \frac{a_1(c-x) + c_1(x-a)}{c-a}, \frac{c_1 - a_1}{c-a}\right)\right| \tag{4}$$

and

$$\left| y'(x) - \frac{c_1 - a_1}{c-a} \right| \leq M. \tag{5}$$

The constant M depends on the rectangle with base $a \leq x \leq c$ and the upper bound of the functions $\alpha(x, y), \beta(x, y)$ in this rectangle; inequality (4) provides that the curve $y = y(x)$ lies in this rectangle.

For the proof of inequality (4) we introduce the following equation [it is a consequence of equation (1)]:

$$y''(x) = F\left(x, \frac{a_1(c-x) + c_1(x-a)}{c-a}, y'(x)\right) + \left[y(x) - \frac{a_1(c-x) + c_1(x-a)}{c-a} \right] F_y(x, \psi, y'(x))$$

in which

$$\psi = \frac{a_1(c-x)+c_1(x-a)}{c-a} + \theta\left[y(x) - \frac{a_1(c-x)+c_1(x-a)}{c-a}\right] \qquad (0<\theta<1).$$

The function

$$y(x) - \frac{a_1(c-x)+c_1(x-a)}{c-a}$$

is equal to zero at $x = a$ and $x = c$. Its maximum absolute value occurs in the interior of the interval (a, c) if it is not identically zero. At the point $x = \xi$, where the maximum absolute value is attained, this function will either have a positive maximum or a negative minimum. Suppose that the former occurs. Then

$$y''(\xi) \leq 0, \qquad y'(\xi) = \frac{c_1 - a_1}{c-a}$$

which implies

$$F\left(\xi, \frac{a_1(c-\xi)+c_1(\xi-a)}{c-a}, \frac{c_1-a_1}{c-a}\right) + k\left[y(\xi) - \frac{a_1(c-\xi)+c_1(\xi-a)}{c-a}\right] \leq 0.$$

Hence

$$y(\xi) - \frac{a_1(c-\xi)+c_1(\xi-a)}{c-a} \leq -\frac{1}{k}F\left(\xi, \frac{a_1(c-\xi)+c_1(\xi-a)}{c-a}, \frac{c_1-a_1}{c-a}\right).$$

The second case follows in a similar manner. This shows that everywhere in $[a, c]$ the following inequality holds:

$$\left|y(x) - \frac{a_1(c-x)+c_1(x-a)}{c-a}\right| \leq \frac{1}{k}\max_{a\leq x\leq c}\left|F\left(x, \frac{a_1(c-x)+c_1(x-a)}{c-a}, \frac{c_1-a_1}{c-a}\right)\right|.$$

Thus inequality (4) has been proved.

In the proof of inequality (5), m denotes the right side of inequality (4) and $\mathfrak{A}$ and $\mathfrak{B}$ the least upper bounds of $\alpha(x, y)$, $\beta(x, y)$ in the rectangle

$$a \leq x \leq c, \qquad |y| \leq m + \max\{|a_1|, |c_1|\}.$$

We assume that $\mathfrak{A} \geq 1$ and introduce the functions u, v by means of the formulas

$$y(x) - \frac{a_1(c - x) + c_1(x - a)}{c - a} + m = \frac{\ln u}{2\mathfrak{A}},$$

$$-y(x) + \frac{a_1(c - x) + c_1(x - a)}{c - a} + m = \frac{\ln v}{2\mathfrak{A}}.$$

Since the left sides of these formulas are not negative in the interval $[a, c]$ it follows that in this interval $u \geq 1$ and $v \geq 1$. Differentiating gives us

$$y'(x) - \frac{c_1 - a_1}{c - a} = \frac{u'}{2\mathfrak{A}u} \tag{6a}$$

$$-y'(x) + \frac{c_1 - a_1}{c - a} = \frac{v'}{2\mathfrak{A}v}. \tag{6b}$$

Differentiating again we find that

$$y''(x) = \frac{u''}{2\mathfrak{A}u} - \frac{u'^2}{2\mathfrak{A}u^2} \tag{7a}$$

$$-y''(x) = \frac{v''}{2\mathfrak{A}v} - \frac{v'^2}{2\mathfrak{A}v^2}. \tag{7b}$$

Inequality (3) allows us to write

$$|y''(x)| \leq 2\mathfrak{A}\left\{y'(x) - \frac{c_1 - a_1}{c - a}\right\}^2 + \mathfrak{B}_1,$$

$$\left[\mathfrak{B}_1 = \mathfrak{B} + 2\mathfrak{A}\left(\frac{c_1 - a_1}{c - a}\right)^2\right]$$

and so

$$y''(x) \geq -2\mathfrak{A}\left\{y'(x) - \frac{c_1 - a_1}{c - a}\right\}^2 - \mathfrak{B}_1$$

and

$$y''(x) \leq 2\mathfrak{A}\left\{y'(x) - \frac{c_1 - a_1}{c - a}\right\}^2 + \mathfrak{B}_1.$$

From equations (6a) and (7a) it follows that

$$\frac{u''}{2\mathfrak{A}u} - \frac{u'^2}{2\mathfrak{A}u^2} \geq -2\mathfrak{A}\frac{u'^2}{4\mathfrak{A}^2u^2} - \mathfrak{B}_1$$

and since $u \geq 1$ we have

$$u'' \geq -2\mathfrak{A}\mathfrak{B}_1 u. \tag{8a}$$

Similarly from equations (6b) and (7b) it follows that

$$v'' \geq -2\mathfrak{A}\mathfrak{B}_1 v. \tag{8b}$$

Let us turn our attention now to the points of the interval (a, c) where

$$y'(x) - \frac{c_1 - a_1}{c - a} \tag{9}$$

vanishes, that is, the points where u' and v' also vanish. There are such points since $u(a) = u(c)$ and $v(a) = v(c)$. These points divide the interval $[a, c]$ into subintervals. Equation (9) ensures that the functions $u'(x)$, $v'(x)$ do not change sign in the subintervals and vanish at one or both of the end points of each subinterval. Let I be one of the subintervals, and let the functions (9), $u'(x)$, and $v'(x)$ be zero at ξ, the right end point. The quantity (9) must be either positive or negative in I.

Suppose it is positive in I. Equation (6a) provides that u' is not negative so if we multiply both sides of equation (8a) by u', we obtain the inequality

$$u''u' \geq -2\mathfrak{A}\mathfrak{B}_1 uu'.$$

Integrating this from $x \in I$ to ξ, and making use of the fact that $u'(\xi) = 0$, we have

$$-u'^2(x) \geq -2\mathfrak{A}\mathfrak{B}_1\{u^2(\xi) - u^2(x)\}.$$

From this it follows that

$$u'^2(x) \leq 2\mathfrak{A}\mathfrak{B}_1 u^2(\xi) \leq 2\mathfrak{A}\mathfrak{B}_1 e^{8\mathfrak{A}m}$$

which implies that

$$\frac{u'^2(x)}{u^2(x)} \leq 2\mathfrak{A}\mathfrak{B}_1 e^{8\mathfrak{A}m};$$

and hence

$$4\mathfrak{A}^2 \left\{ y'(x) - \frac{c_1 - a_1}{c - a} \right\}^2 \leq 2\mathfrak{A}\mathfrak{B}_1 e^{8\mathfrak{A}m}$$

that is, in the interval I

$$\left| y'(x) - \frac{c_1 - a_1}{c - a} \right| \leq \sqrt{\frac{\mathfrak{B}_1}{2\mathfrak{A}}}\, e^{4\mathfrak{A}m}. \tag{10}$$

If the quantity (9) is negative in I, we multiply both sides of inequality (8b) by v'. Equation (6b) provides that v' is not negative. Proceeding as above we again obtain inequality (10) which was to be proved.

c. To prove the theorem, draw an element of the integral curve for which $y'(a) = 0$ through the point $A(a, a_1)$. Take a point $D(d, d_1)$ on it and join it to the point $B(b, b_1)$ by a broken line consisting of two segments parallel to the coordinate axes as in Figure 3. Denote by $P(\xi, \xi_1)$ a variable point on the broken line DQB.

A family of integral curves

$$y = \phi(x, \alpha) \tag{11}$$

can be constructed through the point A where the parameter α represents $y'(a)$. For $\alpha = 0$ the integral curve coincides with AD. If the point P is sufficiently near D we can find a curve of the family (11) which passes through P. In other words if ξ is sufficiently close to d, the equation

$$d_1 = \phi(\xi, \alpha) \tag{12}$$

can be solved for α. There will be only one solution because of uniqueness which has already been proved. Consequently equation (12) defines ξ as a monotonic function of α; this implies α is a monotonic function of ξ. We show that an arbitrary point of DQ can be reached by an integral curve in this way.

Suppose that an arbitrary point of DR other than the point R can thus be reached. When ξ increases to the value of the abscissa

of R the angle α varies monotonically and hence approaches a limit. If this limit is different from $\pm\frac{1}{2}\pi$ the point R is attained; hence $\alpha \to \pm\frac{1}{2}\pi$; that is, as the point P approaches the point R the derivative at the point a of an integral curve $y = y(x)$ joining

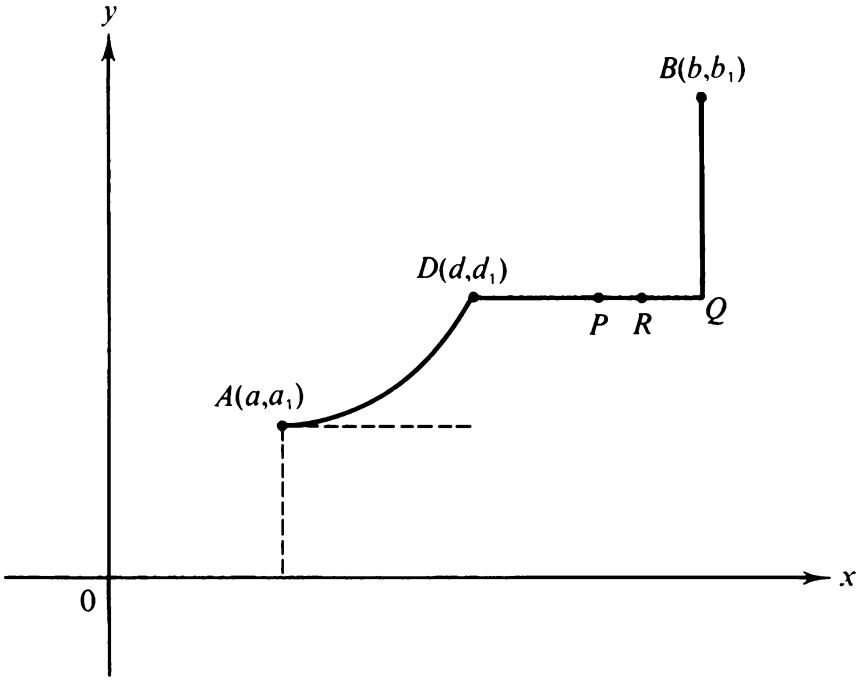

Figure 3

A to P will not remain bounded. This contradicts the evaluation for the derivative of the solution which provides that $|y'(a)|$ cannot be as large as we please because the end points A and P of an integral curve are in a bounded region and the difference of their abscissas does not approach zero; so the point can be reached. Now we let the point P move from Q to B. Here a point P placed close to Q can certainly be reached by an integral curve. In other words, the equation $\xi_1 = \phi(b, \alpha)$ defines α as a single-valued function of ξ_1; again we have the monotonic property. To finish the proof we may repeat the argument that has just been made.

10. THE CALCULUS OF VARIATIONS PROBLEM IN PARAMETRIC FORM

The form of the functional that we have been using is not general enough for geometric applications because a plane curve can be represented by the equation $y = y(x)$ only when every straight line parallel to the y-axis meets the curve in not more than one point. There would be a similar restriction for curves in a space of three or more dimensions. Weierstrass showed how to get rid of this difficulty. We shall present the Weierstrass theory only for the plane case.*

We consider the collection $\mathfrak{M}$ of all piecewise-smooth *trajectories* lying in a given region G of the xy-plane. By trajectory we mean a curve with a definite direction on it. Such a curve is given by two equations

$$x = \phi(t), \qquad y = \psi(t). \tag{1}$$

The parameter t increases from some value t_1 to another value t_2 when the point (x, y) traverses the curve in the designated direction. The functions $\phi(t)$, $\psi(t)$ are continuous with piecewise-continuous derivatives of the first order. At points of continuity both derivatives do not vanish simultaneously; at points of discontinuity of the derivatives neither the left-hand limits nor the right-hand limits are zero simultaneously.

Each such curve has an infinite number of parametric representations. A transition from one parametrization to another

$$x = \tilde{\phi}(\tau), \qquad y = \tilde{\psi}(\tau) \qquad (\tau_1 \leq \tau \leq \tau_2) \tag{1'}$$

is made by transforming the parameter $t = \tilde{\omega}(\tau)$; $\tilde{\omega}(\tau)$ is a function with a piecewise-continuous first derivative that is positive everywhere it exists and has positive limit values at points of discontinuity. Such a transformation of parameter has an inverse $\tau = \omega(t)$; $\omega(t)$ has a positive piecewise-continuous derivative of the first order.

Instead of the function $J[Y]$, which was a functional of a vector

*The basic facts of the theory extend to the case $n > 1$ easily. Some important examples for the case of arbitrary n are found in Sections 19 and 20 of the Appendix.

function, we now consider a function of a line:

$$J_C = \int_C F(t, x, y, \dot{x}, \dot{y})\, dt = \int_{t_1}^{t_2} F(t, x, y, \dot{x}, \dot{y})\, dt.$$

The dots indicate derivatives with respect to the parameter and F is a given function, continuous in all its arguments when (x, y) lies in G; $\dot{x}$, $\dot{y}$ have arbitrary values and are not zero simultaneously; t lies in some finite interval. The function F cannot be completely arbitrary because the integral under consideration must depend only on the curve C and not on any particular parametric representation of the curve; the integral must not change in going from one parametrization (1) to another (1′), that is,

$$\int_{t_1}^{t_2} F(t, \phi(t), \psi(t), \phi'(t), \psi'(t))\, dt = \int_{\tau_1}^{\tau_2} F(\tau, \tilde{\phi}(\tau), \tilde{\psi}(\tau), \tilde{\phi}'(\tau), \tilde{\psi}'(\tau))\, d\tau. \quad (2)$$

Replace τ by $\tau = \omega(t)$ in the right-hand integral. Since

$$\tilde{\phi}(\tau) = \phi(t), \qquad \tilde{\psi}(\tau) = \psi(t),$$

$$\tilde{\phi}'(\tau) = \frac{\phi'(t)}{\omega'(t)}, \qquad \psi'(\tau) = \frac{\psi'(t)}{\omega'(t)},$$

we obtain

$$\int_{\tau_1}^{\tau_2} F(\tau, \tilde{\phi}(\tau), \tilde{\psi}(\tau), \tilde{\phi}'(\tau), \tilde{\psi}'(\tau))\, d\tau = \int_{t_1}^{t_2} F\left(\omega(t), \phi(t), \psi(t), \frac{\phi'(t)}{\omega'(t)}, \frac{\psi'(t)}{\omega'(t)}\right) \omega'(t)\, dt$$

which, with (2), yields

$$\int_{t_1}^{t_2} F(t, \phi(t), \psi(t), \phi'(t), \psi'(t))\, dt = \int_{t_1}^{t_2} F\left(\omega(t), \phi(t), \psi(t), \frac{\phi'(t)}{\omega'(t)}, \frac{\psi'(t)}{\omega'(t)}\right) \omega'(t)\, dt. \quad (3)$$

Because a similar equation must hold for any arbitrary curve in the collection $\mathfrak{M}$ and because an arc of a curve C belongs to both

the collection and the curve, equation (3) must hold if the upper limit t_2 is replaced by an arbitrary value t of the interval (t_1, t_2). If we make this substitution and differentiate with respect to t, we find that

$$F(t, \phi(t), \psi(t), \phi'(t), \psi'(t)) = F\left(\omega(t), \phi(t), \psi(t), \frac{\phi'(t)}{\omega'(t)}, \frac{\psi'(t)}{\omega'(t)}\right)\omega'(t) \quad (4)$$

holds for arbitrary functions $\omega(t)$, $\phi(t)$, $\psi(t)$ that satisfy the conditions assumed above. These conditions are necessary and sufficient for the integral

$$\int_{t_1}^{t_2} F(t, x, y, \dot{x}, \dot{y})\, dt \quad (5)$$

to depend only on the curve and not on its parametrization.

We first set $\omega(t) = t + c$ where c is a constant. Relation (4) becomes

$$F(t, \phi(t), \psi(t), \phi'(t), \psi'(t)) = F(t + c, \phi(t), \psi(t), \phi'(t), \psi'(t)).$$

Because a proper choice of the curve C in $\mathfrak{M}$ and its parametrization (1) makes it possible to assign arbitrary values to $\phi(t)$, $\psi(t)$, $\phi'(t)$, $\psi'(t)$ for any value of t satisfying the conditions

$$(\phi(t), \psi(t)) \in G, \qquad \phi'^2(t) + \psi'^2(t) \neq 0$$

and because the constant c is arbitrary, it follows that t does not appear explicitly in the integrand function F, that is

$$F(t, x, y, \dot{x}, \dot{y}) = F(x, y, \dot{x}, \dot{y}).$$

We return to relation (4) and set $\omega(t) = kt$ in which k is a positive constant. Combined with the relation already obtained, this gives us

$$F(\phi(t), \psi(t), \phi'(t), \psi'(t)) = F\left(\phi(t), \psi(t), \frac{\phi'(t)}{k}, \frac{\psi'(t)}{k}\right)k$$

from which it follows that

$$F(x, y, k\dot{x}, k\dot{y}) = kF(x, y, \dot{x}, \dot{y}).$$

This shows that the integrand $F(x, y, \dot{x}, \dot{y})$ must be positively

homogeneous and of the first degree relative to the second pair of arguments. The conditions obtained are both necessary and sufficient for the validity of relation (4). This implies that the integral (5) depends only on the curve C.

In the construction that follows, it is not enough to assume only continuity for $F = F(x, y, \dot{x}, \dot{y})$; it will be sufficient to assume that F is three times continuously differentiable. Consider some of the consequences of the homogeneity of the function F. If we differentiate both sides of the equation:

$$F(x, y, k\dot{x}, k\dot{y}) = kF(x, y, \dot{x}, \dot{y})$$

with respect to the arbitrary positive number k and then set $k = 1$ we find that

$$F_{\dot{x}}(x, y, \dot{x}, \dot{y})\dot{x} + F_{\dot{y}}(x, y, \dot{x}, \dot{y})\dot{y} = F(x, y, \dot{x}, \dot{y}). \tag{6}$$

Differentiating once with respect to $\dot{x}$ and then with respect to $\dot{y}$, we obtain the equations

$$\dot{x}F_{\dot{x}\dot{x}} + \dot{y}F_{\dot{x}\dot{y}} = 0 \qquad \text{and} \qquad \dot{x}F_{\dot{x}\dot{y}} + \dot{y}F_{\dot{y}\dot{y}} = 0.$$

We conclude from these that

$$\frac{F_{\dot{x}\dot{x}}}{\dot{y}^2} = \frac{F_{\dot{x}\dot{y}}}{-\dot{x}\dot{y}} = \frac{F_{\dot{y}\dot{y}}}{\dot{x}^2}.$$

If we denote the ratio here by F_1 we have

$$F_{\dot{x}\dot{x}} = \dot{y}^2F_1, \qquad F_{\dot{x}\dot{y}} = -\dot{x}\dot{y}F_1, \qquad F_{\dot{y}\dot{y}} = \dot{x}^2F_1. \tag{7}$$

We note that F_1 is positively homogeneous of degree -3 in the second pair of arguments.

To state the problem of a relative extremum for a functional in parametric form we use the notion of strong and weak neighborhood of the curve C. The curve C_1 lies in a strong ϵ-neighborhood of the curve C if parametric representations

$$\begin{aligned} C:&\quad x = \phi(t), \quad y = \psi(t) \\ C_1:&\quad x = \phi_1(t), \quad y = \psi_1(t) \end{aligned} \qquad (t_1 \leq t \leq t_2)$$

exist such that for any $t \in [t_1, t_2]$

$$[\phi_1(t) - \phi(t)]^2 + [\psi_1(t) - \psi(t)]^2 \leq \epsilon^2.$$

The curve C_1 lies in a weak ϵ-neighborhood of the curve C if parametric representations exist that satisfy the preceding inequality and also satisfy

$$[\phi_1'(t) - \phi'(t)]^2 + [\psi_1'(t) - \psi'(t)]^2 \leq \epsilon^2 \sqrt{\phi'^2(t) + \psi'^2(t)}\, \sqrt{\phi_1'^2(t) + \psi_1'^2(t)}$$

at every point t at which the derivatives in the inequality exist. The inequalities which define neighborhoods are invariant relative to admissible transformations of the parameter and have a simple geometric meaning. To wit, a one-to-one, continuous correspondence exists between the set of all points of the curve C and the set of all points of the curve C_1; it provides that the distance of each point of one curve from the corresponding point of the other does not exceed ϵ (first inequality) and that the angle between the tangents to the curves at corresponding points is a quantity of the order of ϵ (second inequality). Let us clarify the second assertion. We designate the angle by χ and write

$$\cos \chi = \frac{\phi'(t)\phi_1'(t) + \psi'(t)\psi_1'(t)}{\sqrt{\phi'^2(t) + \psi'^2(t)}\, \sqrt{\phi_1'^2(t) + \psi_1'^2(t)}}.$$

Hence the second inequality can be expressed in the form

$$\left\{\sqrt[4]{\frac{\phi_1'^2(t) + \psi_1'^2(t)}{\phi'^2(t) + \psi'^2(t)}} - \sqrt[4]{\frac{\phi'^2(t) + \psi'^2(t)}{\phi_1'^2(t) + \psi_1'^2(t)}}\right\}^2 + 2(1 - \cos \chi) \leq \epsilon^2$$

from which

$$1 - \cos \chi \leq \tfrac{1}{2}\epsilon^2$$

and consequently

$$|\chi| \leq \tfrac{1}{2}\pi\epsilon.$$

The basic problem of the calculus of variations for the functionals that we are considering now is this: Find an extremum for the functional J_C in the class of all curves $C \in \mathfrak{M}$ which go from a given point (a_1, b_1) to another given point (a_2, b_2). (This establishes a definite direction on the curves.) We can seek an absolute extremum or a relative extremum (strong or weak).

Suppose an admissible curve

$$\bar{C}: \qquad x = \bar{\phi}(t), \qquad y = \bar{\psi}(t) \qquad (t_1 \leq t \leq t_2)$$

gives the functional J_C a weak relative extremum. The functions $\bar{\phi}(t), \bar{\psi}(t)$ must satisfy the Euler equations

$$F_{\dot{x}} - \int_{t_1}^{t} F_x \, dt = A \tag{8a}$$

$$F_{\dot{y}} - \int_{t_1}^{t} F_y \, dt = B \tag{8b}$$

in which A and B are constants. For the proof it is sufficient to construct the family of admissible curves

$$x = \bar{\phi}(t) + \epsilon\xi(t), \qquad y = \bar{\psi}(t) + \epsilon\eta(t)$$

in which $\xi(t)$ and $\eta(t)$ are arbitrary piecewise-smooth functions that vanish at the ends of the interval $[t_1, t_2]$. We may repeat verbatim all the considerations in Section 3.

Following Section 3 we may derive an equation in addition to equations (8a) and (8b):

$$F - \dot{x}F_{\dot{x}} - \dot{y}F_{\dot{y}} - \int_{t_1}^{t} F_t \, dt = \text{const.}$$

But $F_t \equiv 0$ and this equation becomes

$$F - \dot{x}F_{\dot{x}} - \dot{y}F_{\dot{y}} = 0.$$

In the class of twice differentiable functions $x(t)$, $y(t)$ equations (8a) and (8b) lead to differential equations of the second order. One of these equations must be dependent on the other; if they were independent they would define not only the curve but also its parametrization. This would eliminate the possibility of having more than one parametrization.

To prove dependence, differentiate equation (8a) twice with respect to t. This yields the expression

$$F_{\dot{x}x}\dot{x} + F_{\dot{x}y}\dot{y} + F_{\dot{x}\dot{x}}\ddot{x} + F_{\dot{x}\dot{y}}\ddot{y} = F_x.$$

Taking into account the result (7) and obtaining from equation (6) the equation

$$F_x = F_{xx}\dot{x} + F_{x\dot{y}}\dot{y},$$

we can express it in the form

$$\dot{y}\{(F_{\dot{x}y} - F_{x\dot{y}}) + F_1(\ddot{x}\dot{y} - \ddot{y}\dot{x})\} = 0. \tag{9}$$

Similarly, equation (8b) can be expressed in the form

$$\dot{x}\{(F_{\dot{x}y} - F_{x\dot{y}}) + F_1(\ddot{x}\dot{y} - \ddot{y}\dot{x})\} = 0. \tag{10}$$

Because $\dot{x}(t), \dot{y}(t)$ cannot vanish simultaneously, equations (9) and (10) yield the equation

$$F_{\dot{x}y} - F_{x\dot{y}} + (\ddot{x}\dot{y} - \ddot{y}\dot{x})F_1 = 0. \tag{11}$$

This is called the *Euler-Weierstrass equation.* If $F_1 \not\equiv 0$, it can be written in the form

$$\frac{F_{\dot{x}y} - F_{x\dot{y}}}{F_1(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

The left member is a positively homogeneous function of degree zero in $\dot{x}$ and $\dot{y}$, and the right member represents the curvature of the curve. Hence neither this equation nor the Euler-Weierstrass equation will change if the parameter is changed.

If we use arc length s as parameter, the equations

$$\dot{x} = \frac{dx}{ds} = \cos\theta, \qquad \dot{y} = \frac{dy}{ds} = \sin\theta$$

hold where θ is the angle between the tangent to the curve and the x-axis. Along each smooth arc of a curve

$$\frac{d}{ds}F_{\dot{x}} - F_x = 0, \qquad \frac{d}{ds}F_{\dot{y}} - F_y = 0 \tag{8'}$$

and at a corner point $s = s_0$

$$F_{\dot{x}}\Big|_{s=s_0-0}^{s=s_0+0} = 0, \qquad F_{\dot{y}}\Big|_{s=s_0-0}^{s=s_0+0} = 0. \tag{12}$$

A smooth solution of equations (8') is called an *extremal.* Equations (12) express the *Weierstrass-Erdmann corner condition.* They are equivalent in form, as might have been expected, for a parametric problem.

The notion of singular and nonsingular elements for functionals in parametric form (given below) is similar to that given in Section 6 for functionals in ordinary form. We must take into account that the determinant

$$\begin{vmatrix} F_{\dot{x}\dot{x}} & F_{\dot{x}\dot{y}} \\ F_{\dot{y}\dot{x}} & F_{\dot{y}\dot{y}} \end{vmatrix}$$

is identically zero because of the homogeneity of the function F. Along with this determinant it is convenient to use the function F_1 in connection with the following easily verified relation involving a third-order determinant:

$$F_1(\dot{x}^2 + \dot{y}^2)^2 = -\begin{vmatrix} F_{\dot{x}\dot{x}} & F_{\dot{x}\dot{y}} & \dot{x} \\ F_{\dot{y}\dot{x}} & F_{\dot{y}\dot{y}} & \dot{y} \\ \dot{x} & \dot{y} & 0 \end{vmatrix}. \tag{13}$$

Now we introduce Hilbert's theorem for the functional in parametric form: *If* $x = \bar{x}(s)$, $y = \bar{y}(s)$ *is an extremal for the functional*

$$J_C = \int_C F(x, y, \dot{x}, \dot{y})\, dt$$

and if

$$F_1(\bar{x}(s), \bar{y}(s), \bar{x}'(s), \bar{y}'(s)) \neq 0$$

at some point $s = s_0$, *then there is an interval* $(s_0 - h, s_0 + h)$ *in which* $\bar{x}(s)$ *and* $\bar{y}(s)$ *have continuous derivatives of the second order.*

PROOF: Consider the system of three equations:

$$F_{\dot{x}}(\bar{x}(s), \bar{y}(s), u, v) - lu = \int_0^s F_x(\bar{x}(s), \bar{y}(s), \bar{x}'(s), \bar{y}'(s))\, ds + A \tag{14a}$$

$$F_{\dot{y}}(\bar{x}(s), \bar{y}(s), u, v) - lv = \int_0^s F_y(\bar{x}(s), \bar{y}(s), \bar{x}'(s), \bar{y}'(s))\, ds + B \tag{14b}$$

$$u^2 + v^2 = 1 \tag{14c}$$

in which A and B are the constants of the Euler equations (8).

In some interval, for arbitrary choice of s, this system has a solution

$$u = \bar{x}'(s), \qquad v = \bar{y}'(s), \qquad l = 0.$$

The Jacobian of the left members of equations (14a), (14b), and (14c) with respect to u, v, l is equal to

$$-2\begin{vmatrix} F_{\dot{x}\dot{x}} - l & F_{\dot{x}\dot{y}} & u \\ F_{\dot{y}\dot{x}} & F_{\dot{y}\dot{y}} - l & v \\ u & v & 0 \end{vmatrix}.$$

Formula (13) provides that at the point $u = \bar{x}'(s), v = \bar{y}'(s)$, $l = 0$ this Jacobian is equal to $2F_1$; hence, for $s = s_0$, it is different from zero. This implies that it is different from zero for $s_0 - h < s < s_0 + h$. A theorem on implicit functions implies that consequently the functions $u = \bar{x}'(s), v = \bar{y}'(s), l = 0$ defined by equations (14a), (14b), and (14c) have continuous derivatives for $s_0 - h < s < s_0 + h$.

An *element* in the case of a functional in parametric form is a system x_0, y_0, θ_0, with the system of four numbers $x_0, y_0, \dot{x}_0, \dot{y}_0$ $(\dot{x}_0^2 + \dot{y}_0^2 = 1)$; θ_0 is the angle for which $\cos\theta_0 = \dot{x}_0$ and $\sin\theta_0 = \dot{y}_0$. An element is *nonsingular* if $F_1(x_0, y_0, \cos\theta_0, \sin\theta_0) \neq 0$.

A functional

$$J_C = \int_C F(x, y, \dot{x}, \dot{y})\, dt$$

is *regular* at a point $(x_0, y_0) \in G$ if for arbitrary values of θ

$$F_1(x_0, y_0, \cos\theta, \sin\theta) \neq 0.$$

It is *quasi-regular** if $F_1(x_0, y_0, \cos\theta, \sin\theta)$ may be zero but does not change sign. (In what follows we shall usually assume that the sign is positive.)

In concluding this chapter, we point out that the problem of the calculus of variations in ordinary form is just as important as the problem in parametric form. They are different problems, and our choice of one or the other depends on whether we desire a function or a curve.

---

*The notions of regularity and quasi-regularity have not been introduced for the functional in ordinary form. This will be done in Chapter 2; see Section 15.

EXERCISES

Find extremals for the following functionals:

1. $\displaystyle\int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx$

2. $\displaystyle\int_{x_0}^{x_1} \sqrt{y(1+y'^2)}\,dx$

3. $\displaystyle\int_{x_0}^{x_1} y'(1+x^2y')\,dx$

4. $\displaystyle\int_{x_0}^{x_1} (y'^2+2yy'-16y^2)\,dx$

5. $\displaystyle\int_{x_0}^{x_1} (xy'+y'^2)\,dx$

6. $\displaystyle\int_{x_0}^{x_1} \frac{1+y^2}{y'^2}\,dx$

7. $\displaystyle\int_{x_0}^{x_1} (y^2+y'^2-2y\sin x)\,dx$

8. $\displaystyle\int_{x_0}^{x_1} (2yz-2y^2+y'^2-z'^2)\,dx$

9. $\displaystyle\int_{x_0}^{x_1} (y'^2+z'^2+y'z')\,dx$

10. $\displaystyle\int_{x_0}^{x_1} \frac{y'^2}{x^3}\,dx.$

Investigate the following functionals for extrema:

11. $\displaystyle\int_{x_0}^{x_1} (y^2+2xyy')\,dx, \qquad y(x_0)=y_0, \qquad y(x_1)=y_1.$

12. $\displaystyle\int_{0}^{1} (xy+y^2-2y^2y')\,dx, \qquad y(0)=1, \qquad y(1)=2.$

2

Theory of Fields

11. FIELDS FOR FUNCTIONALS IN ORDINARY FORM

We can now begin to derive sufficient conditions for extrema and also further necessary conditions. The first step is to introduce the notion of a field for the functional

$$J[Y] = \int_a^b f(x, Y, Y')\,dx = \int_a^b f(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')\,dx. \quad (1)$$

We retain all the assumptions prescribed for the functional (1) in Section 1 as well as the notation used there.

We define a *field* for the functional (1) to be any region $D \subset G$ with a vector function

$$P(x, Y) = \{p_1(x, y_1, y_2, \cdots, y_n), \cdots, p_n(x, y_1, y_2, \cdots, y_n)\}$$

that satisfies the following two conditions:

A. The functions $p_i(x, y_1, y_2, \cdots, y_n)$ have continuous partial derivatives of the first order in D.

B. The *Hilbert invariant integral*

$$I_C = \int_C \left[f(x, Y, P) - \sum_{i=1}^n p_i f_{y_i'}(x, Y, P)\right]dx + \sum_{i=1}^n f_{y_i'}(x, Y, P)\,dy_i \quad (2)$$

depends only on the end points of the curve along which it is taken and not on the curve itself. The region D is said to be *covered* by a field and the vector function P is called the *slope function* of the field.

First we find the *trajectories* of the field. They are defined by a system of first-order differential equations:

$$\frac{dy_i}{dx} = p_i(x, y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n).$$

An existence theorem for ordinary differential equations provides that through each point of the region D there passes one and only one trajectory and that the collection of all trajectories is an n-parameter family:

$$y_i = \phi_i(x, \beta_1, \beta_2, \cdots, \beta_n) \qquad (i = 1, 2, \cdots, n). \tag{3}$$

We write this more briefly as

$$Y = \Phi(x, B)$$

in which Φ represents a vector function which, with Φ_x, is continuously differentiable with respect to all the β_i in some region R of the space of the variables $x, \beta_1, \beta_2, \cdots, \beta_n$.

Next we examine the Hilbert invariant integral. Since this integral is independent of path, the expression under the integral sign must be an exact differential, and is governed by the following conditions. There are $n(n-1)/2$ conditions of the form

$$\frac{\partial}{\partial y_i} f_{y_k'}(x, Y, P) = \frac{\partial}{\partial y_k} f_{y_i'}(x, Y, P) \qquad (i, k = 1, 2, \cdots, n) \tag{4}$$

(these conditions are absent for $n = 1$) and n conditions of the form

$$\frac{\partial}{\partial y_k}\left\{f(x, Y, P) - \sum_{i=1}^{n} p_i f_{y_i'}(x, Y, P)\right\} = \frac{\partial}{\partial x} f_{y_k'}(x, Y, P)$$
$$(k = 1, 2, \cdots, n). \tag{5}$$

Carrying out the differentiation, we can write condition (5) in the form

$$f_{y_k} + \sum_{i=1}^{n} f_{y_i'} \frac{\partial p_i}{\partial y_k} - \sum_{i=1}^{n} \frac{\partial p_i}{\partial y_k} f_{y_i'} - \sum_{i=1}^{n} p_i \frac{\partial}{\partial y_k} f_{y_i'}$$
$$= f_{y_k' x} + \sum_{i=1}^{n} f_{y_k' y_i'} \frac{\partial p_i}{\partial x}$$

and if we take condition (4) into account

$$f_{y_k} = f_{y'_k x} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial y_i} f_{y'_k} + \sum_{i=1}^{n} f_{y'_k y'_i} \frac{\partial p_i}{\partial x}.$$

Because

$$\frac{\partial}{\partial y_i} f_{y'_k} = f_{y'_k y_i} + \sum_{j=1}^{n} f_{y'_k y'_j} \frac{\partial p_j}{\partial y_i}$$

this result can be put in the form

$$f_{y_k} = f_{y'_k x} + \sum_{i=1}^{n} f_{y'_k y_i} p_i + \sum_{i=1}^{n} f_{y'_k y'_i} \left(\frac{\partial p_i}{\partial x} + \sum_{j=1}^{n} \frac{\partial p_i}{\partial y_j} p_j \right). \quad (6)$$

Along a trajectory

$$\frac{dy_i}{dx} = p_i.$$

This implies

$$\frac{d^2 y_i}{dx^2} = \frac{\partial p_i}{\partial x} + \sum_{j=1}^{n} \frac{\partial p_i}{\partial y_j} p_j$$

and relation (6) becomes

$$f_{y_k} = f_{y'_k x} + \sum_{i=1}^{n} f_{y'_k y_i} \frac{dy_i}{dx} + \sum_{i=1}^{n} f_{y'_k y'_i} \frac{d^2 y_i}{dx^2}$$

or

$$f_{y_k} - \frac{d}{dx} f_{y'_k} = 0.$$

This last relation shows that the trajectories of the field are extremals.

The family of trajectories of the field is thus an n-parameter family of extremals of the functional. Because the collection of all extremals of the functional is a $2n$-parameter family, the collection of all trajectories must be a proper part of it. Here we ask what conditions an n-parameter family of extremals of the functional must satisfy if there exists a field for the functional such that the considered family of extremals is the family of all trajectories. We shall derive these conditions in Section 12. Note that along each trajectory arc (extremal) of the field, the Hilbert integral reduces to the functional (1) taken along this arc.

12. CONSTRUCTION OF A FIELD

We take an n-parameter family of extremals

$$y_i = \phi_i(x, \beta_1, \beta_2, \cdots, \beta_n) \qquad (i = 1, 2, \cdots, n) \tag{1}$$

which we denote more briefly as

$$Y = \Phi(x, B).$$

Suppose that this family *simply covers* the region D in the space of the variables $x, y_1, y_2, \cdots, y_n$; that is, through every point of this region there passes one and only one extremal of the family. Thus the system (1) defines a point set R in the space of the variables $x, \beta_1, \beta_2, \cdots, \beta_n$ which these formulas transform into the region D. Suppose that R is a simply connected region* and that the functions ϕ_i are twice continuously differentiable in this region.† In addition, assume that

$$\frac{\partial(\phi_1, \phi_2, \cdots, \phi_n)}{\partial(\beta_1, \beta_2, \cdots, \beta_n)} \neq 0 \tag{2}$$

everywhere in the region R.

If we solve the system (1) for β_i, $(i = 1, 2, \cdots, n)$ we obtain twice continuously differentiable functions in the region D:

$$\beta_i = \psi_i(x, y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n) \tag{3}$$

or

$$B = \Psi(x, Y).$$

Through each point $(x, y_1, y_2, \cdots, y_n)$ of the region D there passes one and only one extremal of the family (1). To find its slope at a point we differentiate formulas (1) with respect to x and at the same time eliminate $\beta_1, \beta_2, \cdots, \beta_n$ by using formula (3). We obtain the expression

$$Y' = \Phi_x(x, B) = \Phi_x(x, \Psi(x, Y)).$$

---

*A region is said to be *simply connected* if every closed curve lying in it remains in it when it is reduced to a point by means of a continuous deformation.

†However, we do not need the derivatives $\phi_{i\beta_k\beta_j}$ for $k = j = 1, 2, \cdots, n$.

Now we formulate the problem that is considered in the remaining part of this section: Find the necessary and sufficient conditions that a region D together with a vector function

$$P(x, Y) = \Phi_x(x, \Psi(x, Y)) \tag{4}$$

be a field for our functional. Since the function (4) is continuously differentiable in D, the problem reduces to the following: Find conditions under which the Hilbert integral formed with the aid of the vector function $P(x, Y)$ is independent of path.

We express the Hilbert integral in terms of the variables $x, \beta_1, \beta_2, \cdots, \beta_n$ and a simple calculation making use of the fact that

$$P(x, \Phi(x, B)) = \Phi_x(x, B)$$

yields a new expression for the Hilbert integral:

$$I_C = \int_L f(x, \Phi(x, B), \Phi_x(x, B))\, dx + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} f_{y_i'} \phi_{i\beta_k} \right) d\beta_k. \tag{5}$$

In this integral, L is the path in the region R corresponding to the path C in the region D. If the integral (5) is to depend only on the end points of the path, it is necessary that the expression under the integral sign be an exact differential; since the region is simply connected, this necessary condition is also sufficient. Thus the desired conditions reduce to equations of the form

$$\frac{\partial}{\partial \beta_k} f(x, \Phi, \Phi_x) = \frac{\partial}{\partial x} \sum_{i=1}^{n} f_{y_i'} \phi_{i\beta_k} \qquad (k = 1, 2, \cdots, n) \tag{6}$$

with the $n(n-1)/2$ equations of the form

$$\frac{\partial}{\partial \beta_r} \left(\sum_{i=1}^{n} f_{y_i'} \phi_{i\beta_s} \right) = \frac{\partial}{\partial \beta_s} \left(\sum_{i=1}^{n} f_{y_i'} \phi_{i\beta_r} \right) \qquad \begin{pmatrix} r, s = 1, 2, \cdots, n, \\ r \neq s \end{pmatrix}. \tag{7}$$

By carrying out the differentiation, we can write equations (6) in the form

$$\sum_{i=1}^{n} f_{y_i} \phi_{i\beta_k} + \sum_{i=1}^{n} f_{y_i'} \phi_{ix\beta_k} = \sum_{i=1}^{n} f_{y_i'} \phi_{i\beta_k x} + \sum_{i=1}^{n} \phi_{i\beta_k} \frac{\partial}{\partial x} f_{y_i'}.$$

This implies

$$\sum_{i=1}^{n} \phi_{i\beta_k} \left\{ f_{y_i} - \frac{d}{dx} f_{y_i'} \right\} = 0.$$

We have replaced the symbol $\partial/\partial x$ with the symbol d/dx by taking the quantities β_k, $(k = 1, 2, \cdots, n)$ as constants. The above condition is satisfied automatically because the curve (1) is an extremal.

If we carry out the differentiation in equation (7) we obtain

$$\sum_{i=1}^{n} \phi_{i\beta_s} \frac{\partial}{\partial \beta_r} f_{y_i'} = \sum_{i=1}^{n} \phi_{i\beta_r} \frac{\partial}{\partial \beta_s} f_{y_i'}. \tag{7'}$$

We express this condition in another form by adjoining to y_i $(i = 1, 2, \cdots, n)$ the canonical variables

$$v_i = f_{y_i'}(x, Y, Y') \qquad (i = 1, 2, \cdots, n).$$

With the canonical variables expressed in terms of x and B by means of formula (1), we obtain an n-parameter family of functions

$$v_i = f_{y_i'}(x, \Phi(x, B), \Phi_x(x, B)) \qquad (i = 1, 2, \cdots, n). \tag{8}$$

Formula (1) and equation (8) allow us to write equation (7') as

$$\sum_{i=1}^{n} \left(\frac{\partial y_i}{\partial \beta_s} \frac{\partial v_i}{\partial \beta_r} - \frac{\partial y_i}{\partial \beta_r} \frac{\partial v_i}{\partial \beta_s} \right) = 0 \qquad (r, s = 1, 2, \cdots, n). \tag{9}$$

The left side of equation (9) is called the *Lagrange bracket*, and is denoted by the symbol

$$[\beta_s, \beta_r].$$

We have thus solved the problem we set ourselves. For a given functional let there be given an n-parameter family of extremals with the properties described at the beginning of this section. Then a necessary and sufficient condition that this family be the family of all trajectories of a field for this functional is this: *All the Lagrange brackets, expressed in terms of the n-parameter family* (1), (8) *of canonical variables by means of the corresponding family of extremals* (1), *vanish identically.*

An n-parameter family of extremals (1) for which every Lagrange bracket vanishes is called a *Mayer family*. Thus an n-

parameter family of extremals that satisfies the conditions enumerated at the beginning of this section generates a field if and only if it is a Mayer family. For $n = 1$, every one-parameter family of extremals is a Mayer family.

The following theorem is useful for determining whether or not an n-parameter family of extremals is a Mayer family.

THEOREM OF LAGRANGE: *If the quantities*

$$y_1, y_2, \cdots, y_n, v_1, v_2, \cdots, v_n$$

that depend on x *and the parameters* $\beta_1, \beta_2, \cdots, \beta_n$ *satisfy the canonical equations*

$$\frac{dy_k}{dx} = -\frac{\partial H}{\partial v_k}, \qquad \frac{dv_k}{dx} = \frac{\partial H}{\partial y_k} \qquad (k = 1, 2, \cdots, n),$$

then all Lagrange brackets expressed in these terms of these quantities are independent of x. An important consequence of this theorem is that it is sufficient to verify the vanishing of a Lagrange bracket at only one point x.

We prove the theorem by differentiating:

$$\begin{aligned}
\frac{d}{dx}[\beta_s, \beta_r] &= \sum_{i=1}^{n}\left(\frac{\partial v_i}{\partial \beta_r}\frac{\partial}{\partial \beta_s}\frac{dy_i}{dx} + \frac{\partial y_i}{\partial \beta_s}\frac{\partial}{\partial \beta_r}\frac{dv_i}{dx}\right) \\
&\qquad - \sum_{i=1}^{n}\left(\frac{\partial v_i}{\partial \beta_s}\frac{\partial}{\partial \beta_r}\frac{dy_i}{dx} + \frac{\partial y_i}{\partial \beta_r}\frac{\partial}{\partial \beta_s}\frac{dv_i}{dx}\right) \\
&= \sum_{i=1}^{n}\left\{-\frac{\partial^2 H}{\partial v_i\,\partial \beta_s}\frac{\partial v_i}{\partial \beta_r} + \frac{\partial^2 H}{\partial y_i\,\partial \beta_r}\frac{\partial y_i}{\partial \beta_s}\right. \\
&\qquad \left. + \frac{\partial^2 H}{\partial v_i\,\partial \beta_r}\frac{\partial v_i}{\partial \beta_s} - \frac{\partial^2 H}{\partial y_i\,\partial \beta_s}\frac{\partial y_i}{\partial \beta_r}\right\} \\
&= -\sum_{i=1}^{n}\left(\frac{\partial^2 H}{\partial \beta_s\,\partial v_i}\frac{\partial v_i}{\partial \beta_r} + \frac{\partial^2 H}{\partial \beta_s\,\partial y_i}\frac{\partial y_i}{\partial \beta_r}\right) \\
&\qquad + \sum_{i=1}^{n}\left(\frac{\partial^2 H}{\partial \beta_r\,\partial y_i}\frac{\partial y_i}{\partial \beta_s} + \frac{\partial^2 H}{\partial \beta_r\,\partial v_i}\frac{\partial v_i}{\partial \beta_s}\right) \\
&= -\frac{\partial^2 H}{\partial \beta_s\,\partial \beta_r} + \frac{\partial^2 H}{\partial \beta_r\,\partial \beta_s} = 0.
\end{aligned}$$

EXAMPLE: If all the extremals (1) go through one point $(c, c_1, \cdots, c_n)$ which, of course, cannot be outside of D, then the quantities

$$\phi_i(c, B) = c_i \qquad (i = 1, 2, \cdots, n)$$

do not depend on $\beta_1, \beta_2, \cdots, \beta_n$. Hence

$$\left.\frac{\partial y_i}{\partial \beta_r}\right|_{x=c} = \frac{\partial \phi_i(c, B)}{\partial \beta_r} = 0 \qquad (i, r = 1, 2, \cdots, n).$$

This implies that all the Lagrange brackets vanish at $x = c$. The theorem that has just been proved implies that the Lagrange brackets are identically zero and that the family of extremals is a Mayer family. Consequently this family generates a so-called *central field.*

13. FIELDS FOR FUNCTIONALS IN PARAMETRIC FORM

The definition of a *field* for the functional in parametric form

$$J_C = \int_C F(x, y, \dot{x}, \dot{y})\, dt$$

is analogous to that for a functional in ordinary form. A region $D \in G$ of the xy-plane with a function $\theta = \theta(x, y)$ that is continuously differentiable in D and is called the *slope of the field* constitutes a field if the Hilbert integral

$$I_L = \int_L F_{\dot{x}}(x, y, \cos\theta, \sin\theta)\, dx + F_{\dot{y}}(x, y, \cos\theta, \sin\theta)\, dy \quad (1)$$

does not depend on the path of integration L in D but only on the initial and terminal points of the path.

A trajectory of the field is defined as an integral curve of the system

$$\frac{dx}{ds} = \cos\theta, \qquad \frac{dy}{ds} = \sin\theta.$$

The condition that the integral (1) be independent of the path is:

$$F_{\dot{x}y} + F_{\dot{x}\dot{x}}\frac{\partial p}{\partial y} + F_{\dot{x}\dot{y}}\frac{\partial q}{\partial y} = F_{\dot{y}x} + F_{\dot{y}\dot{x}}\frac{\partial p}{\partial x} + F_{\dot{y}\dot{y}}\frac{\partial q}{\partial x} \quad (2)$$

in which

$$p = \cos\theta, \qquad q = \sin\theta.$$

If we recall the function F_1, defined in Section 10, we can write relation (2) as follows:

$$F_{\dot{x}y} - F_{\dot{y}x} + F_1\left\{q^2\frac{\partial p}{\partial y} - pq\frac{\partial q}{\partial y} + pq\frac{\partial p}{\partial x} - p^2\frac{\partial q}{\partial x}\right\} = 0. \quad (2')$$

The trajectories must satisfy this relation. But along a trajectory

$$\frac{d^2x}{ds^2} = \frac{\partial p}{\partial x}p + \frac{\partial p}{\partial y}q, \qquad \frac{d^2y}{ds^2} = \frac{\partial q}{\partial x}p + \frac{\partial q}{\partial y}q.$$

This implies that, along a trajectory, relation (2′) has the form

$$F_{\dot{x}y}(x(s), y(s), x'(s), y'(s)) - F_{x\dot{y}}(x(s), y(s), x'(s), y'(s)) + F_1(x(s), y(s), x'(s), y'(s))\{y'(s)x''(s) - x'(s)y''(s)\} = 0.$$

This equation shows that trajectories are extremals of the functional.

Suppose a one-parameter family of extremals

$$\begin{aligned} x &= \phi(t, \beta) \\ y &= \psi(t, \beta) \end{aligned} \qquad (t_1 \leq t \leq t_2, \beta_1 \leq \beta \leq \beta_2) \quad (3)$$

simply covers a simply connected region D of the xy-plane. If the functions $\phi(t, \beta)$, $\psi(t, \beta)$, $\phi_t(t, \beta)$, $\psi_t(t, \beta)$ are continuously differentiable in the rectangular region R, $(t_1 \leq t \leq t_2, \beta_1 \leq \beta \leq \beta_2)$ and if

$$\frac{\partial(\phi, \psi)}{\partial(t, \beta)} \neq 0$$

in this region, then the family (3) defines a field.

The proof of this fact is omitted; we merely call attention to the fact that the slope of the field can be found with the aid of formula (3). First, equations (3) can be solved almost everywhere for t and β:

$$t = t(x, y), \qquad \beta = \beta(x, y).$$

These two functions have continuous first-order derivatives in the

region D. All that remains is to express the quantities

$$\cos\theta = \frac{\phi_t(t, \beta)}{\sqrt{[\phi_t(t, \beta)]^2 + [\psi_t(t, \beta)]^2}},$$

$$\sin\theta = \frac{\psi_t(t, \beta)}{\sqrt{[\phi_t(t, \beta)]^2 + [\psi_t(t, \beta)]^2}}$$

in terms of x and y by means of our formulas.

14. THE WEIERSTRASS E-FUNCTION

The Weierstrass E-function for the functional in ordinary form

$$J[Y] = \int_a^b f(x, Y, Y')\, dx$$

is defined as the following function of the variable x and the three vectors Y, Z, Q:

$$E(x, Y, Z, Q) = f(x, Y, Q) - f(x, Y, Z) - \sum_{i=1}^{n} (q_i - z_i) f_{y_i'}(x, Y, Z). \quad (1)$$

If we think of f as a function of its last vector argument, then we see that $E(x, Y, Z, Q)$ is the difference between the value of the function f at the point Q and the linear part of its Taylor expansion about the point Z. Hence

$$E(x, Y, Z, Q) = \frac{1}{2!} \sum_{i,k=1}^{n} (q_i - z_i)(q_k - z_k) f_{y_i' y_k'}(x, Y, Z + \theta(Q - Z)) \quad (2)$$

in which $0 < \theta < 1$.

The Weierstrass E-function for the functional in parametric form

$$J_C = \int_C F(x, y, \dot{x}, \dot{y})\, dt$$

is defined as

$$\begin{aligned}\mathfrak{E}(x, y, \theta, \phi) &= E(x, y, \cos\theta, \sin\theta, \cos\phi, \sin\phi)\\ &= F(x, y, \cos\phi, \sin\phi) - F(x, y, \cos\theta, \sin\theta)\\ &\quad - (\cos\phi - \cos\theta)F_{\dot{x}}(x, y, \cos\theta, \sin\theta)\\ &\quad - (\sin\phi - \sin\theta)F_{\dot{y}}(x, y, \cos\theta, \sin\theta).\end{aligned}$$

This is similar to the E-function for the functional in ordinary form. However, the homogeneity of the function F allows us to simplify this expression. Because of the properties of F, we can write

$$\begin{aligned}&F(x, y, \cos\theta, \sin\theta)\\ &\quad = \cos\theta \cdot F_{\dot{x}}(x, y, \cos\theta, \sin\theta) + \sin\theta \cdot F_{\dot{y}}(x, y, \cos\theta, \sin\theta).\end{aligned}$$

Thus

$$\begin{aligned}&\mathfrak{E}(x, y, \theta, \phi)\\ &\quad = F(x, y, \cos\phi, \sin\phi) - \cos\phi \cdot F_{\dot{x}}(x, y, \cos\theta, \sin\theta)\\ &\qquad\qquad - \sin\phi \cdot F_{\dot{y}}(x, y, \cos\theta, \sin\theta)\\ &\quad = \cos\phi\{F_{\dot{x}}(x, y, \cos\phi, \sin\phi) - F_{\dot{x}}(x, y, \cos\theta, \sin\theta)\}\\ &\quad + \sin\phi\{F_{\dot{y}}(x, y, \cos\phi, \sin\phi) - F_{\dot{y}}(x, y, \cos\theta, \sin\theta)\}. \qquad (3)\end{aligned}$$

For a further transformation we use the identity

$$\begin{aligned}&\frac{d}{d\tau} F_{\dot{x}}(x, y, \cos\tau, \sin\tau)\\ &\quad = -\sin\tau F_{\dot{x}\dot{x}}(x, y, \cos\tau, \sin\tau) + \cos\tau F_{\dot{x}\dot{y}}(x, y, \cos\tau, \sin\tau)\\ &\quad = -\sin\tau F_1(x, y, \cos\tau, \sin\tau).\end{aligned}$$

Integration of this from θ to ϕ yields

$$\begin{aligned}&F_{\dot{x}}(x, y, \cos\phi, \sin\phi) - F_{\dot{x}}(x, y, \cos\theta, \sin\theta)\\ &\qquad\qquad = -\int_{\theta}^{\phi} F_1(x, y, \cos\tau, \sin\tau)\sin\tau\, d\tau.\end{aligned}$$

Next we derive the analogous expression:

$$\begin{aligned}&F_{\dot{y}}(x, y, \cos\phi, \sin\phi) - F_{\dot{y}}(x, y, \cos\theta, \sin\theta)\\ &\qquad\qquad = \int_{\theta}^{\phi} F_1(x, y, \cos\tau, \sin\tau)\cos d\tau.\end{aligned}$$

If we use these formulas, we can write equation (3) in the form

$$\mathfrak{E}(x, y, \theta, \phi) = \int_{\theta}^{\phi} F_1(x, y, \cos \tau, \sin \tau) \sin (\phi - \tau)\, d\tau. \quad (4)$$

To construct the function $\mathfrak{E}(x, y, \theta, \phi)$ for a given θ, we can alter ϕ in an arbitrary interval of length 2π. If we take such an interval $[\theta - \pi, \theta + \pi]$, we find that

$$-\pi \leq \phi - \theta \leq \pi.$$

Hence, in the integral (4), one of the inequalities:

$$0 \leq \phi - \tau \leq \pi,$$

$$-\pi \leq \phi - \tau \leq 0$$

holds for all τ in the interval of integration. In both cases, the function $\sin (\phi - \tau)$ does not change sign in the interval of integration; consequently the first mean-value theorem applies. It gives:

$$\begin{aligned} \mathfrak{E}(x, y, \theta, \phi) &= F_1(x, y, \cos \theta^*, \sin \theta^*) \int_{\theta}^{\phi} \sin (\phi - \tau)\, d\tau \\ &= F_1(x, y, \cos \theta^*, \sin \theta^*)\{1 - \cos (\phi - \theta)\}. \quad (5) \end{aligned}$$

Thus we see that

$$F_1(x, y, \cos \theta, \sin \theta) = \lim_{\phi \to \theta} \frac{\mathfrak{E}(x, y, \theta, \phi)}{1 - \cos (\phi - \theta)}.$$

15. SUFFICIENT CONDITIONS FOR A STRONG OR A WEAK MINIMUM FOR A FUNCTIONAL IN ORDINARY FORM

Suppose that $Y = \bar{Y}(x)$ $(a \leq x \leq b)$ is an extremal joining the points (a, A) and (b, B) for the functional

$$J[Y] = \int_a^b f(x, Y, Y')\, dx. \quad (1)$$

We say that the extremal $Y = \bar{Y}(x)$ can be *imbedded in a field* if a field exists that covers some region containing the extremal $Y = \bar{Y}(x)$ and has this extremal $Y = \bar{Y}(x)$ as one of its trajectories. In such a case the following assertions are true:

A. *If an extremal* $Y = \bar{Y}(x)$ *can be imbedded in a field and if, in some region (call it* D*) that contains the extremal* $Y = \bar{Y}(x)$ *and is covered by the field, the inequality*

$$E(x, Y, P(x, Y), Q) \geq 0 \tag{2}$$

holds for arbitrary finite values of the vector Q *and for* $P(x, Y)$ *(the slope function of the field) then* $Y = \bar{Y}(x)$ *furnishes the functional* (1) *with a strong minimum.*

B. *If an extremal* $Y = \bar{Y}(x)$ *can be imbedded in a field and if*

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x, \bar{Y}, \bar{Y}')\rho_i\rho_k > 0 \qquad (a \leq x \leq b) \tag{3}$$

for arbitrary values of ρ_i *that satisfy the condition*

$$\sum_{i=1}^{n} \rho_i^2 \neq 0,$$

then the extremal $Y = \bar{Y}(x)$ *furnishes a weak minimum.*

For $n = 1$, condition (3) has the form

$$f_{y'y'}(x, \bar{y}, \bar{y}') > 0 \qquad (a \leq x \leq b).$$

Legendre discovered the importance of both this inequality and the weakened inequality

$$f_{y'y'}(x, \bar{y}, \bar{y}') \geq 0 \qquad (a \leq x \leq b).$$

Accordingly, the inequality

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x, \bar{Y}, \bar{Y}')\rho_i\rho_k \geq 0 \qquad (a \leq x \leq b)$$

is known as the *Legendre condition* and inequality (3), as the *strengthened Legendre condition.** Inequality (2) is called the *Weierstrass condition.* (Sometimes it is called the Weierstrass condition for a field.)

To prove proposition A, consider a piecewise-smooth curve $Y = \tilde{Y}(x)$ (designated as $\tilde{L}$) that lies in the region D and satisfies the boundary conditions. Along an extremal $Y = \bar{Y}(x)$ which we

---

*We dwell on Legendre's method as well as his results below.

denote by $\bar{L}$

$$J[\bar{Y}] = I_{\bar{L}}.$$

Also, the Hilbert integral does not depend on the path of integration. Thus we may write

$$J[\bar{Y}] = I_{\bar{L}} = \int_a^b \Big\{ f(x, \tilde{Y}, P(x, \tilde{Y})) - \sum_{i=1}^{n} [p_i(x, \tilde{Y}) - \tilde{y}_i'] f_{y_i'}(x, \tilde{Y}, P(x, \tilde{Y})) \Big\} dx.$$

This implies that

$$J[\tilde{Y}] - J[\bar{Y}] = \int_a^b \Big\{ f(x, \tilde{Y}, \tilde{Y}') - f(x, \tilde{Y}, P(x, \tilde{Y})) + \sum_{i=1}^{n} [p_i(x, \tilde{Y}) - \tilde{y}_i'] f_{y_i'}(x, \tilde{Y}, P(x, \tilde{Y})) \Big\} dx$$

$$= \int_a^b E(x, \tilde{Y}, P(x, \tilde{Y}), \tilde{Y}')\, dx.$$

The Weierstrass condition (2) guarantees that the right member is not negative, which proves that

$$J[\tilde{Y}] \geq J[\bar{Y}].$$

Turning to the second proposition, we consider the real quadratic form

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x, Y, Z) \rho_i \rho_k$$

and introduce its minimum $\mu(x, Y, Z)$ on the hypersphere $\rho_1^2 + \rho_2^2 + \cdots + \rho_n^2 = 1$.

The function $\mu(x, Y, Z)$ is continuous in all its arguments. We note that for arbitrary real ρ_i,

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x, Y, Z) \rho_i \rho_k \geq \mu(x, Y, Z) \sum_{i=1}^{n} \rho_i^2. \tag{4}$$

By hypothesis we have

$$\mu(x, \bar{Y}(x), \bar{Y}'(x)) \geq \mu > 0 \qquad (a \leq x \leq b).$$

It follows from this and from the continuity of the function $\mu(x, Y, Z)$ that for sufficiently small $\delta > 0$

$$\mu(x, R(x), S(x)) \geq \frac{\mu}{2} \qquad (a \leq x \leq b) \tag{5}$$

whenever

$$\begin{aligned} &\|R(x) - \bar{Y}(x)\| \leq \delta, \\ &\|S(x) - \bar{Y}'(x)\| \leq \delta \qquad (a \leq x \leq b). \end{aligned} \tag{6}$$

Let us take a positive number $\epsilon \leq \delta$ sufficiently small so that every curve $Y = \tilde{Y}(x)$ $(a \leq x \leq b)$ for which

$$\|\tilde{Y}(x) - \bar{Y}(x)\| \leq \epsilon \qquad (a \leq x \leq b)$$

lies in the region D and satisfies the inequality

$$\|P(x, \tilde{Y}) - \bar{Y}'(x)\| \leq \delta \qquad (a \leq x \leq b).$$

Such a quantity $\epsilon = \epsilon(\delta)$ can be found since $\bar{Y}'(x) = P(x, \bar{Y})$ which implies that the required inequality has the form

$$\|P(x, \tilde{Y}) - P(x, \bar{Y})\| \leq \delta \qquad (a \leq x \leq b).$$

The validity of this last inequality for sufficiently small $\epsilon > 0$ follows from the continuity of the function $P(x, Y)$.

To prove proposition B, take an arbitrary smooth curve

$$Y = \tilde{Y}(x) \qquad (a \leq x \leq b)$$

that lies in a weak ϵ-neighborhood of the curve $Y = \bar{Y}(x)$ $(a \leq x \leq b)$ and satisfies the given boundary conditions. Then, as above, we have

$$J[\tilde{Y}] - J[\bar{Y}] = \int_a^b E(x, \tilde{Y}, P(x, \tilde{Y}), \tilde{Y}')\, dx.$$

But by formula (2) in Section 14

$$\begin{aligned} &E(x, \tilde{Y}, P(x, \tilde{Y}), \tilde{Y}') \\ &= \tfrac{1}{2} \sum_{i,k=1}^{n} [\tilde{y}_i' - p_i(x, \tilde{Y})][\tilde{y}_k' - p_k(x, \tilde{Y})] f_{y_i' y_k'}(x, R(x), S(x)) \end{aligned}$$

in which

$$R(x) = \tilde{Y}(x),\ S(x) = P(x, \tilde{Y}) + \theta[\tilde{Y}'(x) - P(x, \tilde{Y})] \qquad (0 < \theta < 1).$$

Condition (6) is satisfied because

$$\|R(x) - \bar{Y}(x)\| = \|\tilde{Y}(x) - \bar{Y}(x)\| \leq \epsilon \leq \delta \qquad (a \leq x \leq b)$$

and

$$\begin{aligned}\|S(x) - \bar{Y}'(x)\| &= \|P(x, \tilde{Y}) - \bar{Y}'(x) + \theta[\tilde{Y}'(x) - P(x, \tilde{Y})]\| \\ &= \|(1 - \theta)[P(x, \tilde{Y}) - \bar{Y}'(x)] + \theta[\tilde{Y}'(x) - \bar{Y}'(x)]\| \\ &\leq (1 - \theta)\delta + \theta\epsilon \leq \delta.\end{aligned}$$

Consequently inequality (5) is satisfied, and relation (4) provides that the inequality

$$E(x, \tilde{Y}, P(x, \tilde{Y}), \tilde{Y}') \geq \tfrac{1}{2} \sum_{k=1}^{n} [\tilde{y}_k' - p_k(x, \tilde{Y})]^2 \frac{\mu}{2} \geq 0$$

is satisfied on the whole interval $[a, b]$. It follows that

$$J[\tilde{Y}] \geq J[\bar{Y}]$$

and proposition B has been proved.

The criteria of this section demonstrate the advantage of having a class of functionals for which*

$$\mu(x, Y, Z) > 0$$

for an arbitrary finite vector Z at an arbitrary point $(x, Y) \in G$ or the weakened inequality

$$\mu(x, Y, Z) \geq 0.$$

In the first case, the functionals are called *regular* and, in the second case, *quasi-regular*. The Weierstrass condition (2) is automatically satisfied for quasi-regular functionals.

*The definition given applies to minimum problems. For a maximum, the symbol $<$ replaces $>$.

16. SUFFICIENT CONDITIONS FOR A STRONG OR A WEAK MINIMUM FOR A FUNCTIONAL IN PARAMETRIC FORM

Suppose that the curve

$$\bar{C}: \qquad x = \bar{x}(s), \qquad y = \bar{y}(s) \qquad (s_1 \leq s \leq s_2)$$

is an extremal for the functional

$$J_C = \int_{t_1}^{t_2} F(x, y, \dot{x}, \dot{y})\, dt. \tag{1}$$

Suppose also that this extremal joins two given points and that it can be imbedded in a field with slope function $\theta(x, y)$. The following assertions are valid:

A. *If, for arbitrary* ϕ

$$\mathfrak{E}(x, y, \theta(x, y), \phi) \geq 0 \tag{2}$$

in some region D *that contains an extremal* $\bar{C}$ *and is covered by a field, then the extremal* $\bar{C}$ *gives the functional* J_C *a strong minimum.**

B. *If*

$$F_1(\bar{x}(s), \bar{y}(s), \bar{x}'(s), \bar{y}'(s)) > 0 \qquad (s_1 \leq s \leq s_2) \tag{3}$$

(the strengthened Legendre condition in Weierstrass form) then the extremal $\bar{C}$ *furnishes a weak minimum.*

The proof of these propositions is omitted because it is analogous to the proof of the corresponding propositions for functionals in ordinary form.

In contrast to the case of the functional in ordinary form, proposition (A) for a functional in parametric form leads to another result — one in which the inequality that appears involves only the extremal itself and not the extremals of the field in which it is imbedded. This is the proposition:

A*. *If an extremal* $\bar{C}$ *can be imbedded in a field and if for arbitrary values of* ϕ,

$$F_1(\bar{x}(s), \bar{y}(s), \cos\phi, \sin\phi) > 0$$

---

*Condition (2) is always fulfilled if the functional is quasi-regular. See the definition in Section 10.

at every point of the extremal, then the extremal $\bar{C}$ gives the functional (1) *a strong minimum.*

PROOF: The quantity

$$\min_{0 \leq \tau \leq 2\pi} F_1(x, y, \cos \tau, \sin \tau) = \mu(x, y)$$

is a continuous function of the point $(x, y) \in D$; by hypothesis,

$$\mu(\bar{x}, \bar{y}) \geq \mu$$

for all points $(\bar{x}, \bar{y})$ of the extremal $\bar{C}$; here μ is some positive constant. This means that we can find $\epsilon > 0$ such that

$$\mu(x, y) \geq \frac{\mu}{2} \tag{4}$$

at any point $(x, y) \in D$ whose distance from some point of the extremal $\bar{C}$ is less than or equal to ϵ. If we replace the region D by a region $D_1 \subset D$ that contains the extremal $\bar{C}$, we can be sure that inequality (4) holds at any point $(x, y) \in D_1$. Now consider relation (5) of Section 14; assume that $(x, y) \in D_1$ with θ and ϕ arbitrary. This relation provides that

$$\mathfrak{E}(x, y, \theta, \phi) = F_1(x, y, \cos \theta^*, \sin \theta^*)\{1 - \cos(\phi - \theta)\}$$

$$\geq \frac{\mu}{2}\{1 - \cos(\phi - \theta)\} \geq 0.$$

We see that A* reduces to A.

17. NECESSARY CONDITIONS OF WEIERSTRASS AND LEGENDRE

Having established that certain conditions are sufficient for the verification of an assertion, we want to determine to what extent each of these conditions is necessary.

Let us consider the Weierstrass condition that is expressed by inequality (2) of Section 15 and also the Legendre condition expressed by inequality (3) of the same section. We show that these conditions in a weakened form are necessary for minimizing the

functional

$$J[Y] = \int_a^b f(x, Y, Y')\, dx.$$

Corresponding results can be obtained for functionals in parametric form. Since they are obtained in a similar way we do not discuss them here.

THEOREM 1. THE WEIERSTRASS NECESSARY CONDITION FOR A STRONG MINIMUM: *For an arbitrary finite vector Q the inequality*

$$E(x, Y, Y', Q) \geq 0 \tag{1}$$

must be satisfied at every point of an extremal

$$Y = \bar{Y}(x) \qquad a \leq x \leq b \tag{2}$$

if the extremal (2) *furnishes the given functional with a strong minimum.*

PROOF: Suppose that for some value Q_0 of the vector Q, condition (1) does not hold at some point $x_0 \in [a, b]$, that is,

$$E(x_0, \bar{Y}(x_0), \bar{Y}'(x_0), Q_0) < 0. \tag{3}$$

To be specific, let us suppose that $x_0 > a$. Take $h > 0$ so small that $x_0 - h > a$, and construct the vector function $Y_h(x)$:

$$Y_h(x) = \bar{Y}(x) \qquad \text{for } x_0 \leq x \leq b$$

$$Y_h(x) = Q_0(x - x_0) + \bar{Y}(x_0) \qquad \text{for } x_0 - h \leq x \leq x_0$$

$$Y_h(x) = \bar{Y}(x) + K(x - a) \qquad \text{for } a \leq x \leq x_0 - h$$

in which the vector $K = \{k_1, k_2, \cdots, k_n\}$ is defined by the equation

$$\bar{Y}(x_0 - h) + K(x_0 - a - h) = -Q_0 h + \bar{Y}(x_0). \tag{4}$$

According to the definition, K depends on the quantity h.

For $n = 1$, the curve $Y = Y_h(x)$ is shown in Figure 4.

Our problem is to show that we can make the difference

$$J[Y_h] - J[\bar{Y}] = \Delta(h)$$

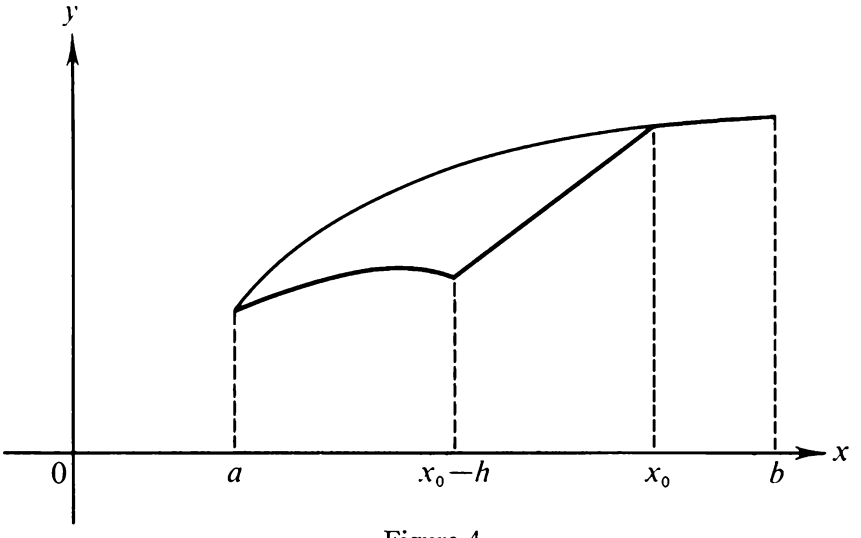

Figure 4

negative by choosing h properly. To do this, we compute $\Delta'(0)$. Because

$$\Delta(h) = \int_a^{x_0-h} f(x, \bar{Y}(x) + K(x-a), \bar{Y}'(x) + K)\, dx$$
$$+ \int_{x_0-h}^{x_0} f(x, \bar{Y}(x_0) + Q_0(x-x_0), Q_0)\, dx$$
$$- \int_a^{x_0} f(x, \bar{Y}(x), \bar{Y}'(x))\, dx,$$

differentiation yields

$$\Delta'(h) = -f(x_0 - h, \bar{Y}(x_0) - Q_0 h, \bar{Y}'(x_0 - h) + K)$$
$$+ \int_a^{x_0-h} \left\{ \sum_{i=1}^{n} f_{y_i}(x, \bar{Y}(x) + K(x-a), \bar{Y}'(x) + K)(x-a) \frac{dk_i}{dh} \right.$$
$$\left. + \sum_{i=1}^{n} f_{y_i'}(x, \bar{Y}(x) + K(x-a), \bar{Y}'(x) + K) \frac{dk_i}{dh} \right\} dx$$
$$+ f(x_0 - h, \bar{Y}(x_0) - Q_0 h, Q_0),$$

and we have

$$\begin{aligned}\Delta'(0) &= f(x_0, \bar{Y}(x_0), Q_0) - f(x_0, \bar{Y}(x_0), \bar{Y}'(x_0)) \\ &\quad + \sum_{i=1}^{n} \left(\frac{dk_i}{dh}\right)_{h=0} \int_a^{x_0} \{f_{y_i}(x, \bar{Y}(x), \bar{Y}'(x))(x - a) \\ &\quad + f_{y_i'}(x, \bar{Y}(x), \bar{Y}'(x))\}\, dx. \end{aligned} \tag{5}$$

Since $\bar{Y}(x)$ is an extremal

$$\begin{aligned}&\int_a^{x_0} \{f_{y_i}(x, \bar{Y}(x), \bar{Y}'(x))(x-a) + f_{y_i'}(x, \bar{Y}(x), \bar{Y}'(x))\}\, dx \\ &= \int_a^{x_0} \left\{(x-a)\frac{d}{dx} f_{y_i'}(x, \bar{Y}(x), \bar{Y}'(x)) + f_{y_i'}(x, \bar{Y}(x), \bar{Y}'(x))\right\} dx \\ &= (x-a) f_{y_i'}(x, \bar{Y}(x), \bar{Y}'(x))\Big|_{x=a}^{x=x_0} \\ &= (x_0-a) f_{y_i'}(x_0, \bar{Y}(x_0), \bar{Y}'(x_0)). \end{aligned} \tag{6}$$

In addition to this, differentiation of equation (4) with respect to h yields

$$-\bar{Y}'(x_0 - h) - K + \frac{dK}{dh}(x_0 - a - h) = -Q_0.$$

If we set $h = 0$ we find that

$$\left(\frac{dK}{dh}\right)_{h=0} (x_0 - a) = -Q_0 + \bar{Y}'(x_0).$$

Relations (5), (6), and (3) produce the inequality

$$\begin{aligned}\Delta'(0) &= f(x_0, \bar{Y}(x_0), Q_0) - f(x_0, \bar{Y}(x_0), \bar{Y}'(x_0)) \\ &\quad - \sum_{i=1}^{n} [q_{0i} - \bar{y}_i'(x_0)] f_{y_i'}(x_0, \bar{Y}(x_0), \bar{Y}'(x_0)) \\ &= E(x_0, \bar{Y}(x_0), \bar{Y}'(x_0), Q_0) < 0. \end{aligned}$$

We know that $\Delta(0) = 0$ and we have shown that $\Delta'(0) < 0$; thus, for sufficiently small $h > 0$, certainly $\Delta(h) < 0$. This proves the theorem. From this theorem we prove

THEOREM 2. THE LEGENDRE NECESSARY CONDITION FOR A WEAK MINIMUM: *The inequality*

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x, Y, Y')\rho_i\rho_k \geq 0 \tag{7}$$

is satisfied at all points of the extremal

$$Y = \bar{Y}(x) \qquad (a \leq x \leq b) \tag{2}$$

for arbitrary values of ρ_i, $(i = 1, 2, \cdots, n)$ *if the extremal* (2) *furnishes a weak minimum for the functional.*

PROOF: Suppose that

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x_0, \bar{Y}(x_0), \bar{Y}'(x_0))\alpha_i\alpha_k < 0, \tag{8}$$

where x_0 is some point of the interval $[a, b]$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are components of some vector A. We express relation (2) of Section 14 in the form

$$E(x_0, \bar{Y}(x_0), \bar{Y}'(x_0), \bar{Y}'(x_0) + tA) = \frac{t^2}{2!}\sum_{i,k=1}^{n} \alpha_i\alpha_k f_{y_i' y_k'}(x_0, \bar{Y}(x_0), \bar{Y}'(x_0) + \theta tA).$$

This relation and equation (8) yield

$$\lim_{t\to 0} \frac{1}{t^2} E(x_0, \bar{Y}(x_0), \bar{Y}'(x_0), \bar{Y}'(x_0) + tA) < 0.$$

Hence, for all $t > 0$ sufficiently small*

$$E(x_0, \bar{Y}(x_0), \bar{Y}'(x_0), \bar{Y}'(x_0) + tA) < 0.$$

Now we set

$$Q_0 = \bar{Y}'(x_0) + tA$$

and consider the curve

$$Y = Y_{h,t}(x) \qquad (a \leq x \leq b) \tag{9}$$

*This inequality is also true for all negative t sufficiently small in absolute value.

that we defined for the proof of Theorem 1 and that depends here on t as well as on h. We find that for sufficienily small $h > 0, t > 0$

$$J[Y_{h,t}] < J[\bar{Y}].$$

On the other hand, for sufficiently small $h > 0, t > 0$, the curve (9) lies in an arbitrarily small weak neighborhood of the extremal (2). This proves the theorem.

18. THE JACOBI CONDITION

In this section we shall derive conditions an extremal must satisfy if it can be imbedded in a field. We consider the problem only for $n = 1$ although the theory can be extended to any $n > 1$.

Given the functional

$$J[y] = \int_a^b f(x, y, y')\, dx$$

with the boundary conditions

$$y(a) = a_1, \qquad y(b) = b_1$$

we suppose an extremal $y = \bar{y}(x)$ furnishes this functional with a weak minimum. As in the derivation of the fundamental necessary condition for extrema, we set

$$y = \bar{y}(x) + \epsilon\eta(x)$$

in which $\eta(x)$ is a piecewise-smooth function that vanishes at the ends of the interval $[a, b]$. By hypothesis, the function

$$\phi(\epsilon) = J[\bar{y} + \epsilon\eta] = \int_a^b f(x, \bar{y} + \epsilon\eta, \bar{y}' + \epsilon\eta')\, dx$$

has a minimum for $\epsilon = 0$. It follows that $\phi'(0) = 0$ and $\phi''(0) \geq 0$. The equation $\phi'(0) = 0$ was used to derive the Euler equation; now we make use of the inequality $\phi''(0) \geq 0$.

First notice that the expression

$$\phi''(0) = \int_a^b (\bar{f}_{yy}\eta^2 + 2\bar{f}_{yy'}\eta\eta' + \bar{f}_{y'y'}\eta'^2)\, dx$$

is a quadratic functional of η. We call it the *accessory functional* and designate it by $I_2[\eta]$. The coefficients of the three terms of the integrand are functions of x. For convenience we set

$$\bar{f}_{yy} = P, \qquad \bar{f}_{yy'} = Q, \qquad \bar{f}_{y'y'} = R.$$

Thus

$$I_2[\eta] = \int_a^b (P\eta^2 + 2Q\eta\eta' + R\eta'^2)\, dx.$$

Legendre is credited with the first study of this accessory functional (1786); he used this expression to derive the condition that bears his name.

To transform the accessory functional, Legendre first introduced an arbitrary continuously differentiable function $w(x)$. Since $\eta(x)$ vanishes at the end points of the interval

$$\int_a^b \frac{d}{dx}(\eta^2 w)\, dx = 0$$

or

$$\int_a^b (\eta^2 w' + 2\eta\eta' w)\, dx = 0.$$

Hence

$$I_2[\eta] = \int_a^b \{(P + w')\eta^2 + 2(Q + w)\eta\eta' + R\eta'^2\}\, dx. \qquad (1)$$

Legendre selected the function $w(x)$ in a definite way. The accessory functional then took the form mentioned above; from this Legendre drew two conclusions:

A. *For an extremal $\bar{y}(x)$ to furnish a minimum (even if weak), it is necessary that $R(x) \geq 0$, $(a \leq x \leq b)$.*

B. *If the stronger condition, $R(x) > 0$, $(a \leq x \leq b)$, is satisfied, the extremal $\bar{y}(x)$ does furnish a weak minimum.*

The first statement is true and is a simple consequence of the theorem proved in Section 17. The second statement is not true; Lagrange pointed out the fallacy in Legendre's proof in 1797. However, a local change in Legendre's statement yields the following valid statement:

B*. *If $R(c) > 0$, $(a \leq x \leq b)$ then we can find a sufficiently*

small subinterval $[\alpha, \beta] \subset [a, b]$ *that contains the point* $x = c$ *so that* $\bar{y}(x)$ *furnishes a minimum for the functional**

$$J^{\alpha\beta}[y] = \int_{\alpha}^{\beta} f(x, y, y')\, dx$$

for

$$y(\alpha) = \bar{y}(\alpha) \equiv \alpha_1, \qquad y(\beta) = \bar{y}(\beta) \equiv \beta_1;$$

if $y(x) \not\equiv \bar{y}(x) \qquad (\alpha \leq x \leq \beta)$, *then*

$$J^{\alpha\beta}[y] > J^{\alpha\beta}[\bar{y}].$$

Because going from a minimum to a maximum corresponds to going from the inequality $R(x) > 0$, $[R(x) \geq 0]$ to the inequality $R(x) < 0$, $[R(x) \leq 0]$, proposition A follows almost immediately from proposition B*. We prove proposition B* now.†

Suppose $R(c) > 0$. Consider the differential equation

$$P + w' = \frac{(Q + w)^2}{R} + \sigma \tag{2}$$

in which σ is a positive constant. The continuity of the function $R(x)$, which is positive at the point $x = c$, provides that the differential equation (2) has a solution $w(x)$ in some interval $[\alpha, \beta]$ that contains the point $x = c$ and belongs to the interval $[a, b]$. We substitute the solution $w(x)$ which has a continuous derivative in the expression for $I_2[\eta]$ after changing the limits a, b to α, β. This yields

$$I_2^{\alpha\beta}[\eta] = \int_{\alpha}^{\beta} \left\{ \sigma\eta^2 + R\left(\eta' + \frac{Q + w}{R}\eta\right)^2 \right\} dx.$$

*Jacobi first pointed out that an arc of a curve that furnishes an extremum if its right-hand end point is near the fixed left-hand end point, can cease to furnish an extremum if the right-hand end point is sufficiently far from the left-hand end point. Investigation of this fact leads to general criteria known as the Jacobi condition.

†To show that proposition A follows from proposition B*, suppose that the function $R(x)$ is negative at some point of the interval $[a, b]$. We assume that proposition B* has been proved; it allows us to change the function $\bar{y}(x)$ in a sufficiently small interval $[\alpha, \beta]$, without changing its values at the ends of this interval, so that the integral decreases in the interval $[\alpha, \beta]$. Without changing the function $\bar{y}(x)$ outside this interval we obtain a function, defined on the whole interval $[a, b]$, which gives the functional a smaller value than that given by the extremal $\bar{y}(x)$.

We compare the value of the functional $J^{\alpha\beta}[y]$ along the extremal $\bar{y}(x)$ with its value along an arbitrary function $y(x)$ that belongs to some weak neighborhood of the extremal and satisfies the conditions

$$y(\alpha) = \alpha_1, \qquad y(\beta) = \beta_1.$$

If we set

$$y(x) - \bar{y}(x) = \eta(x),$$

Taylor's formula yields

$$J^{\alpha\beta}[y] = J^{\alpha\beta}[\bar{y}] + \int_\alpha^\beta (\bar{f}_y\eta + \bar{f}_{y'}\eta')\,dx + \tfrac{1}{2}I_2^{\alpha\beta}[\eta] + \tfrac{1}{6}\int_\alpha^\beta (K\eta^3 + 3L\eta^2\eta' + 3M\eta\eta'^2 + N\eta'^3)\,dx,$$

in which

$$K = f_{yyy}(x, \bar{y} + \theta\eta, \bar{y}' + \theta\eta'), \qquad L = f_{yyy'}(x, \bar{y} + \theta\eta, \bar{y}' + \theta\eta')$$

$$M = f_{yy'y'}(x, \bar{y} + \theta\eta, \bar{y}' + \theta\eta'), \qquad N = f_{y'y'y'}(x, \bar{y} + \theta\eta, \bar{y}' + \theta\eta')$$

$$0 < \theta < 1, \qquad \theta = \theta(x).$$

We set

$$\eta' + \frac{Q + w}{R}\eta = \zeta$$

and then

$$K\eta^3 + 3L\eta^2\eta' + 3M\eta\eta'^2 + N\eta'^3 = A\eta^3 + B\eta^2\zeta + C\eta\zeta^2 + D\zeta^3.$$

If we assume that

$$|\eta| < \epsilon_1, \qquad |\eta'| < \epsilon_1,$$

then the quantities A, B, C, D do not exceed in absolute value some constant Γ which we can estimate in advance. If we consider the expression for $I_2^{\alpha\beta}[\eta]$ and the equation

$$\int_\alpha^\beta (\bar{f}_y\eta + \bar{f}_{y'}\eta')\,dx = 0,$$

we can write

$$J^{\alpha\beta}[y] - J^{\alpha\beta}[\bar{y}] = \tfrac{1}{2}\int_\alpha^\beta (\sigma\eta^2 + R\zeta^2)\,dx$$

$$+ \tfrac{1}{6}\int_\alpha^\beta (A\eta^3 + B\eta^2\zeta + C\eta\zeta^2 + D\zeta^3)\,dx$$

$$\geq \tfrac{1}{2}\int_\alpha^\beta \left\{\eta^2\left(\sigma - \frac{\Gamma}{3}|\eta| - \frac{\Gamma}{3}|\zeta|\right) + \zeta^2\left(R - \frac{\Gamma}{3}|\eta| - \frac{\Gamma}{3}|\zeta|\right)\right\} dx.$$

Now we take a positive number $\epsilon \leq \epsilon_1$ so small that, for $\alpha \leq x \leq \beta$, $|\eta| \leq \epsilon$, $|\eta'| \leq \epsilon$, we have the inequalities

$$\sigma - \frac{\Gamma}{3}(|\eta| + |\zeta|) > 0, \qquad R - \frac{\Gamma}{3}(|\eta| + |\zeta|) > 0.$$

We obtain

$$J^{\alpha\beta}[y] > J^{\alpha\beta}[\bar{y}]$$

whenever $y(x)$ lies in a weak ϵ-neighborhood of the extremal $\bar{y}(x)$ and $y(x) \not\equiv \bar{y}(x)$, $(\alpha \leq x \leq \beta)$. Thus we have proved proposition B*.

Proposition B cannot be proved by this method because equation (2) may not have a continuous solution on the entire interval even though the continuous function $R(x)$ is positive on the entire closed interval $[a, b]$.

To illustrate this, let $P = -1$, $Q = 0$, $R = 1$, so that equation (2) has the form

$$w' = \rho^2 + w^2, \qquad (\rho^2 = 1 + \sigma)$$

in which ρ is a positive constant. Integration of this equation yields

$$w(x) = \rho \tan \rho(x - C)$$

in which C is an arbitrary constant. In every interval of length $\geq \pi/\rho$ the function $w(x)$ becomes infinite.

If, in addition to assuming that $R(x)$ is positive on the interval $[a, b]$, we also assume that the equation

$$P + w' = \frac{(Q + w)^2}{R} \tag{3}$$

has a continuous solution on the entire interval $[a, b]$ then it is easy to show that the extremal $\bar{y}(x)$, $(a \leq x \leq b)$ furnishes the functional with a weak minimum. If equation (3) has a continuous

solution on the whole interval $[a, b]$, then equation (2) has a solution that is continuous over the whole interval $[a, b]$ for all sufficiently small values of $|\sigma|$; the proof can be made for extremals in general. These statements follow from a theorem about the continuity of a solution that depends on a parameter.

In Section 15, we showed that if $R(x)$ is positive and the extremal can be imbedded in a field, a weak minimum is assured. Now we see that if $R(x)$ is positive and equation (3) has a solution which is continuous on the whole interval $[a, b]$, a weak minimum is also assured. Naturally the question arises as to whether these two conditions are equivalent for $R(x) \not\equiv 0$, $(a \leq x \leq b)$. We show that they are equivalent and also investigate the necessity of these conditions.

First we present an example which shows that if the function $R(x)$ is zero even at one point of the interval $[a, b]$ the equivalence that we mentioned does not hold. Suppose

$$J[y] = \int_{-1}^{1} (x^2y'^2 + 12y^2)\,dx, \qquad y(-1) = -1, \qquad y(1) = 1.$$

The general solution of the Euler equation has the form

$$y = C_1x^3 + \frac{C_2}{x^4}.$$

The boundary conditions imply that

$$y = x^3 \equiv \bar{y}(x).$$

This curve cannot be imbedded in a field because the unique one-parameter family of extremals containing the extremal $y = \bar{y}(x)$, $(-1 \leq x \leq 1)$ has the form

$$y = \beta x^3$$

and a region cannot be covered by this family if a point with abscissa $x = 0$ lies in it. In the case under consideration

$$P = 24, \qquad Q = 0, \qquad R = 2x^2,$$

and the Legendre equation (3) has the solution $w = 8x$ which is continuous on the whole interval $[-1, 1]$.

We begin our proof that the above conditions are equivalent by considering the fact that the Legendre equation is a Riccati equation. By making a well-known substitution we can transform this to a second-order linear differential equation. To do this we set

$$\frac{Q + w}{R} = -\frac{u'}{u}.$$

Having defined w from this relation, we differentiate, and substitute for w and w' in equation (3) to obtain the *Jacobi equation*:*

$$-(Ru')' + (P - Q')u = 0. \tag{4}$$

The fundamental problem with respect to the Legendre equation (3) is whether it has a solution that is continuous in the closed interval $[a, b]$. This leads to the question of whether solutions of the Jacobi equation vanish in the closed interval $[a, b]$.†

We call attention to the fact that the Jacobi equation is the Euler equation for the accessory functional. We note also that there is a close connection between the Jacobi equation and the Euler equation

$$\frac{d}{dx} f_{y'} - f_y = 0 \tag{5}$$

for the original functional

$$J[y] = \int_a^b f(x, y, y')\, dx.$$

The Jacobi equation appears as an *equation in the variations* relative to the Euler equation. To clarify this statement, suppose that

$$y = g(x, \alpha), \qquad (a \leq x \leq b, \alpha_1 \leq \alpha \leq \alpha_2)$$

is a one-parameter family of solutions of equation (5) such that

*We must verify that the functions Q and R are continuously differentiable. This follows from Hilbert's theorem, Section 6, which applies here because $R > 0$, $(a \leq x \leq b)$, which in turn implies that $\bar{y}(x)$ has a continuous second derivative.

†If u is not identically zero, but vanishes at some point, then uniqueness theorems for differential equations provide that $u' \neq 0$. This implies that $w = \infty$.

$g(x, \alpha)$ and $g_x(x, \alpha)$ are continuously differentiable functions of α; set

$$g(x, \alpha_0) \equiv \bar{y}(x), \qquad (a \leq x \leq b, \alpha_1 < \alpha_0 < \alpha_2).$$

Then the function

$$g_\alpha(x, \alpha_0) = u(x), \qquad (a \leq x \leq b)$$

is a solution of the Jacobi equation (4) constructed for the extremal $\bar{y}(x)$.

We add the following proposition: Consider that

$$y = g(x, \alpha, \beta), \qquad (a \leq x \leq b, \alpha_1 \leq \alpha \leq \alpha_2, \beta_1 \leq \beta \leq \beta_2)$$

is a two-parameter family of solutions of (5); the functions $g(x, \alpha, \beta)$ and $g_x(x, \alpha, \beta)$ are continuously differentiable functions of α and β, and

$$\frac{\partial(g, g_x)}{\partial(\alpha, \beta)} \neq 0, \qquad (a \leq x \leq b, \alpha_1 \leq \alpha \leq \alpha_2, \beta_1 \leq \beta \leq \beta_2). \quad (6)$$

If

$$g(x, \alpha_0, \beta_0) \equiv \bar{y}(x), \qquad (a \leq x \leq b, \alpha_1 < \alpha_0 < \alpha_2, \beta_1 < \beta_0 < \beta_2)$$

then the functions

$$u_1(x) = g_\alpha(x, \alpha_0, \beta_0), \qquad u_2(x) = g_\beta(x, \alpha_0, \beta_0), \qquad (a \leq x \leq b)$$

represent a pair of linearly independent solutions of the Jacobi equation.

Linear independence follows from condition (6) if the remaining conditions have already been satisfied. We derive these conditions by replacing y in equation (5) with $g(x, \alpha)$ or $g(x, \alpha, \beta)$ and the derivatives obtained by differentiating with respect to the parameter α or β.

The symbol $\Delta(x, c)$ designates a nontrivial solution* of the Jacobi equation that vanishes at $x = c$. The symbol c^* designates the zero of the function $\Delta(x, c)$ nearest to $x = c$ on the right, if, in general, such a zero exists. We call this point *conjugate* to the point c. We also call the two points of the extremal $y = \bar{y}(x)$ with abscissas c and c^* *conjugate points.*

*The theory of differential equations provides that this solution is defined uniquely to within a constant multiplier.

Now we come to the theorems that answer the questions we proposed.

THEOREM 1: *If $R(x) > 0$ at every point of the interval $[a, b]$ and if the point a^* conjugate to a lies in the interval (a, b), the extremal $\bar{y}(x)$ will not furnish a minimum (not even a weak minimum).*

PROOF: Suppose that the extremal $\bar{y}(x)$ furnishes the functional $J[y]$ with a weak relative minimum. Then the accessory functional is positive or zero for every admissible function $\eta(x)$. Hence every piecewise-smooth function that vanishes at $x = a$ and $x = b$ and for which the functional $I_2[\eta]$ vanishes must consist of arcs of extremals of this functional, and at points of discontinuity it must satisfy the Weierstrass-Erdmann corner condition for the functional $I_2[\eta]$. Consider the function

$$\eta_0(x) = \begin{cases} \Delta(x, a), & (a \leq x \leq a^*) \\ 0, & (a^* \leq x \leq b). \end{cases}$$

For this function we have

$$\begin{aligned} I_2[\eta_0] &= \int_a^{a^*} (R\eta_0'^2 + 2Q\eta_0\eta_0' + P\eta_0^2)\, dx \\ &= \int_a^{a^*} \{(P\eta_0 + Q\eta_0')\eta_0 + (Q\eta_0 + R\eta_0')\eta_0'\}\, dx \\ &= \int_a^{a^*} \eta_0 \left\{(P\eta_0 + Q\eta_0') - \frac{d}{dx}(Q\eta_0 + R\eta_0')\right\} dx \\ &= \int_a^{a^*} \eta_0\{-(R\eta_0')' + (P - Q')\eta_0\}\, dx = 0. \end{aligned}$$

Hence the function $\eta_0(x)$ must satisfy the Weierstrass-Erdmann corner condition which implies that

$$(R\eta_0' + Q\eta_0)\Big|_{x=a^*-0}^{x=a^*+0} = 0$$

or

$$R(a^*)\{\eta_0'(a^* + 0) - \eta_0'(a^* - 0)\} = 0.$$

But $R(a^*) > 0$ and $\eta_0'(a^* + 0) = 0$. It follows that $\eta_0'(a^* - 0) = 0$.

However, this is absurd because

$$\eta_0'(a^* - 0) = \Delta_x(a^*, a)$$

and the derivative $\Delta_x(x, a)$ vanishes at the point $x = a^*$ where the function $\Delta(x, a)$ itself vanishes; uniqueness theorems for differential equations provide that this is impossible since $\Delta(x, a)$ is not identically zero.

The requirement that the point conjugate to the left-hand end point of the interval (a, b) cannot lie in the interval is called the *Jacobi condition.*

Theorem 1 states that *if the strengthened Legendre condition is satisfied along an extremal then, for a minimum (even a weak minimum), it is necessary that the extremal satisfy the Jacobi condition.*

THEOREM 2: *If $R(x) > 0$ at every point of the interval $[a, b]$ and if there is no point conjugate to a in the closed interval $[a, b]$ then the extremal $\bar{y}(x)$ can be imbedded in a field and it thus furnishes the functional with a weak minimum. It is assumed that the curve $y = \bar{y}(x)$ lies wholly in the region G for $a \leq x \leq b$.*

We note that the Jacobi condition appears here in *strengthened* form. We note also that if we put aside the particular case in which $a^* = b$ or $R(x)$ vanishes (but does not change sign) then it follows from Theorem 1 and Theorem 2 that it is necessary for an extremum that the extremal be imbedded in a field.

PROOF: The curve $y = \bar{y}(x)$, $(a \leq x \leq b)$ lies entirely in the region G, thus it can be prolonged a little to the right or left with the preservation of its property of being an extremal. We must find a point c, $(c < a)$, sufficiently close to $x = a$ that in the closed interval $[c, b]$ there is no point conjugate to the point c.

We use a theorem of Sturm to prove that this can be done; it states that between any two zeros of one solution of the equation

$$-(Ru')' + (P - Q')u = 0$$

lies a zero of any other solution of this equation. We construct a solution of the Jacobi equation $\Delta(x, b)$ that equals zero at the right-hand end point of the interval. This function is different from zero in the half-closed interval $[a, b)$. Indeed, if it vanishes for

$x = a$, then it coincides with $\Delta(x, a)$ up to a constant factor, and by hypothesis, the function $\Delta(x, a)$ is not zero for $x = b$. Similarly, the function cannot vanish at any point $x = g$ between a and b, $(a < g < b)$ because Sturm's theorem insures that $\Delta(x, a)$ would have a zero between g and b.

Since the function $\Delta(x, b)$ is continuous, it is different from zero in some larger interval $[a_0, b)$, $(a_0 < a)$. We take a point c that satisfies the inequality $a_0 < c < a$ and we show that there is no point conjugate to c in the closed interval $[c, b]$, that is, c is the point that we were seeking. The equality $c^* = b$ is excluded immediately because otherwise it would follow that $\Delta(x, c) = \text{const} \cdot \Delta(x, b)$. On the other hand, Sturm's theorem provides that $c^* < b$ is also impossible because there would be a zero of $\Delta(x, b)$ in the interval (c, c^*) and hence in (a_0, b).

Now we draw a pencil of extremals of our functional through the point with coordinates $x = c$, $y = \bar{y}(c) = c_1$; we use the slope of the extremal at the vertex of the pencil as the parameter. The existence of the pencil follows from a theorem on imbedding an integral curve in a family (see Section 8). Let the pencil be

$$y = g(x, \alpha).$$

A theorem on the dependence of a solution of a differential equation on parameters (see Section 8) provides that both $g(x, \alpha)$ and $g_x(x, \alpha)$ are continuously differentiable with respect to α.

Suppose $\alpha = \alpha_0$ defines the extremal. By hypothesis,

$$\Delta(x, c) \neq 0, \qquad (c < a_0 \leq x \leq b_0 < c^*), \qquad (a_0 < a, b < b_0);$$

thus

$$g_\alpha(x, \alpha_0) \neq 0, \qquad (a_0 \leq x \leq b_0)$$

and continuity provides that

$$g_\alpha(x, \alpha) \neq 0, \qquad (a_0 \leq x \leq b_0, |\alpha - \alpha_0| \leq \epsilon). \tag{7}$$

It follows from this inequality that no two extremals of the family

$$y = g(x, \alpha), \qquad (a_0 \leq x \leq b_0, |\alpha - \alpha_0| \leq \epsilon) \tag{8}$$

have a common point in the strip $a_0 \leq x \leq b_0$. If there were a

value of $x \in [a_0, b_0]$ for which

$$g(x, \alpha_1) = g(x, \alpha_2), \qquad (\alpha_0 - \epsilon \leq \alpha_1 < \alpha_2 \leq \alpha_0 + \epsilon)$$

then, by Rolle's theorem, the function $g_\alpha(x, \alpha)$ would vanish for some value of α in the interval (α_1, α_2); this would contradict inequality (7). On the basis of the general considerations stated in Section 12 the family (8) generates a field in which $y = \bar{y}(x)$ is imbedded. This proves Theorem 2.

We conclude the present section with a geometric interpretation of conjugate point. Let us take a point (c, c_1) on the extremal

$$y = \bar{y}(x) \equiv g(x, \alpha_0). \tag{9}$$

A point of this extremal that is conjugate to (c, c_1) has abscissa c^* and ordinate $g(c^*, \alpha_0)$ where

$$\Delta(c^*, c) = 0.$$

This equation can be written in the form

$$g_\alpha(c^*, \alpha_0) = 0.$$

Thus the coordinates of the conjugate point are defined by the system of equations

$$y = g(x, \alpha_0)$$

$$0 = g_\alpha(x, \alpha_0).$$

By adjoining the equation

$$0 = g_\alpha(x, \alpha)$$

to the equation

$$y = g(x, \alpha)$$

of a family of curves without singular points, we obtain a system of equations that defines the *characteristic points* on the curves of the family under consideration. By definition, the geometric locus of characteristic points is the envelope of the family. Thus the point on the extremal (9) conjugate to the point (c, c_1) is the characteristic point of the extremal nearest to (c, c_1) on the right if a pencil of extremals with vertex (c, c_1) is taken as the family. Setting aside the particular case in which the envelope is degenerate or has a singular point, we can formulate the following rule: *To*

get a point conjugate to the point (c, c_1), *we must construct the envelope of a pencil of extremals with vertex* (c, c_1) *and take the first point to the right of* (c, c_1) *on the extremal that is a point of contact of the extremal with the envelope.* An interesting idea for a proof of Theorem 1 is based on this geometric interpretation; this proof must be painstaking because it is necessary to consider exceptional cases.

19. THE HAMILTON-JACOBI THEORY

Consider the functional

$$J[Y] = \int_a^b f(x, Y, Y')\, dx$$

for arbitrary n. Suppose a field for this functional exists and let $P(x, Y)$ be the slope function for the field; let D be a region covered by the field. We assume that at all points of the region D

$$|f_{y_i' y_j'}(x, Y, P(x, Y))|_{i,j=1}^{n} \neq 0. \tag{1}$$

In other words, for $(x, Y) \in D$ the element

$$x, y_1, y_2, \cdots, y_n, p_1(x, Y), \cdots, p_n(x, Y)$$

is always nonsingular relative to the functional. We introduce the Hilbert integral

$$W(x, Y) = \int_{(x_0, Y_0)}^{(x, Y)} \left[f - \sum_{i=1}^{n} p_i(x, Y) f_{y_i'} \right] dx + \sum_{i=1}^{n} f_{y_i'}\, dy_i.$$

This integral is independent of path in D; it depends only on the initial and terminal points. Therefore it represents a function of the coordinates $x, y_1, y_2, \cdots, y_n$ and is obviously twice continuously differentiable. We have

$$\frac{\partial W}{\partial x} = f(x, Y, P(x, Y)) - \sum_{i=1}^{n} p_i(x, Y) f_{y_i'}(x, Y, P(x, Y)), \tag{2}$$

$$\frac{\partial W}{\partial y_i} = f_{y_i'}(x, Y, P(x, Y)), \qquad (i = 1, 2, \cdots, n). \tag{3}$$

It follows from condition (1) that, in a sufficiently small neighbor-

hood of an arbitrary point of D, the system (3) can be solved for $p_i(x, Y)$. If we substitute these values in equation (2), eliminating $p_i(x, Y)$, we get a partial differential equation for W. To write this equation we recall how the canonical variables were introduced in Section (7) and how the Euler equation was put in canonical form. We did this by using the formulas

$$H(x, Y, V) = f(x, Y, Z) - \sum_{i=1}^{n} z_i f_{y_i'}(x, Y, Z),$$

$$v_i = f_{y_i'}(x, Y, Z), \qquad (i = 1, 2, \cdots, n)$$

in which the right-hand members are identical with the right-hand members of formulas (2) and (3) if the vector $P(x, Y)$ is replaced by Z. Because the derivatives $\partial W/\partial y_i$, $(i = 1, 2, \cdots, n)$ play the role of the quantities v_i, the desired partial differential equation has the form

$$\frac{\partial W}{\partial x} = H\left(x, y_1, y_2, \cdots, y_n, \frac{\partial W}{\partial y_1}, \frac{\partial W}{\partial y_2}, \cdots, \frac{\partial W}{\partial y_n}\right).$$

This equation is known as the *Hamilton-Jacobi equation.*

Each field has its own function W, but all possible functions W satisfy in this way one and only one first-order partial differential equation.

We start from this equation. Supposing that it has an integral, let us examine its significance for our problem in the calculus of variations.

THEOREM 1: *Suppose that, in a region D, there exists a twice continuously differentiable solution $W = W(x, Y)$ of the Hamilton-Jacobi equation and suppose, for $(x, Y) \in D$, that an element*

$$x, y_1, y_2, \cdots, y_n, \frac{\partial W}{\partial y_1}, \frac{\partial W}{\partial y_2}, \cdots, \frac{\partial W}{\partial y_n}$$

lies in a region of xYV-space which the formulas

$$x = x, \qquad y_i = y_i, \qquad v_i = f_{y_i'}(x, Y, Z), \qquad (i = 1, 2, \cdots, n),$$

map in a one-to-one manner on a region of xYZ-space consisting

exclusively of nonsingular elements. Then there exists a field in the region D and, except for an additive constant, $W(x, Y)$ represents the value of the Hilbert integral with variable upper limit for this field.

PROOF: Consider the functions

$$p_i(x, Y) = -H_{v_i}\left(x, y_1, y_2, \cdots, y_n, \frac{\partial W}{\partial y_1}, \frac{\partial W}{\partial y_2}, \cdots, \frac{\partial W}{\partial y_n}\right), \qquad (i = 1, 2, \cdots, n). \quad (4)$$

Our assumptions provide that they are continuously differentiable; let us show that they are the components of the slope of some field. To do this we must verify that the Hilbert integral expressed in terms of these functions is independent of path. To construct the Hilbert integral, we must first find the functions

$$f_{y_i'}(x, Y, P(x, Y)), \qquad (i = 1, 2, \cdots, n).$$

For this purpose we notice that the mapping formulas transform

$$v_i = f_{y_i'}(x, Y, Z), \qquad (i = 1, 2, \cdots, n)$$

to

$$z_i = -H_{v_i}(x, Y, V), \qquad (i = 1, 2, \cdots, n).$$

But this becomes relation (4) if we set

$$z_i = p_i(x, Y), \qquad v_i = \frac{\partial W}{\partial y_i}, \qquad (i = 1, 2, \cdots, n).$$

By hypothesis, the conversion of relation (4) can be completed; it leads to the equations

$$\frac{\partial W}{\partial y_i} = f_{y_i'}(x, Y, P(x, Y)), \qquad (i = 1, 2, \cdots, n).$$

It is also necessary to find the function

$$f(x, Y, P(x, Y)) - \sum_{i=1}^{n} p_i(x, Y) f_{y_i'}(x, Y, P(x, Y)) = H\left(x, y_1, y_2, \cdots, y_n, \frac{\partial W}{\partial y_1}, \frac{\partial W}{\partial y_2}, \cdots, \frac{\partial W}{\partial y_n}\right).$$

The Hamilton-Jacobi equation provides that this function is equal to $\partial W/\partial x$. The preceding remarks imply that the Hilbert integral has the form

$$\int \frac{\partial W}{\partial x}\,dx + \sum_{i=1}^{n} \frac{\partial W}{\partial y_i}\,dy_i.$$

This completes the proof of Theorem 1.

THEOREM 2: *Suppose the function*

$$W(x, Y, B) = W(x, y_1, y_2, \cdots, y_n, \beta_1, \beta_2, \cdots, \beta_n)$$

that depends on a constant vector

$$B = \{\beta_1, \beta_2, \cdots, \beta_n\}$$

satisfies the Hamilton-Jacobi equation

$$\frac{\partial W}{\partial x} = H\left(x, y_1, y_2, \cdots, y_n, \frac{\partial W}{\partial y_1}, \frac{\partial W}{\partial y_2}, \cdots, \frac{\partial W}{\partial y_n}\right) \tag{5}$$

in some neighborhood of the point

$$(x_0, Y_0) = (x_0, y_{01}, y_{02}, \cdots, y_{0n})$$

for all values of the constant vector B in some neighborhood of the vector

$$B_0 = \{\beta_{01}, \beta_{02}, \cdots, \beta_{0n}\}.$$

If the function W is continuously differentiable with respect to the parameters β_i, $(i = 1, 2, \cdots, n)$ and if each of the partial derivatives $\partial W/\partial \beta_i$, $\partial W/\partial y_i$, $(i = 1, 2, \cdots, n)$ is continuously differentiable with respect to all the arguments where

$$\left| \frac{\partial^2 W}{\partial y_j \partial \beta_k} \right|_{k,j=1}^{n} \neq 0, \tag{6}$$

then the partial derivatives

$$\frac{\partial W}{\partial y_j} = v_j, \qquad (j = 1, 2, \cdots, n) \tag{7a}$$

and

$$\frac{\partial W}{\partial \beta_k} = \alpha_k, \qquad (k = 1, 2, \cdots, n) \tag{7b}$$

where $A = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ *is a second arbitrary constant vector, represent a general integral of the canonical system*

$$\frac{dy_j}{dx} = -\frac{\partial H}{\partial v_j}, \qquad \frac{dv_j}{dx} = \frac{\partial H}{\partial y_j}, \qquad (j = 1, 2, \cdots, n) \tag{8}$$

in some neighborhood of the point x_0.

This canonical system is completely defined by the given Hamilton-Jacobi equation; we consider it here independently of any problem in the calculus of variations. Originally we supposed only that the function H had continuous derivatives with respect to the variables y_i, v_i, $(i = 1, 2, \cdots, n)$. If we also assume that the function H belongs to some nonsingular functional, then Theorem 2 provides that we obtain a general integral of the Euler equations for this functional.

PROOF: If we substitute the values x_0, Y_0, B_0 for x, Y, B in the left members of equations (7a) and (7b), we obtain the two vectors

$$V_0 = \{v_{01}, v_{02}, \cdots, v_{0n}\}, \qquad A_0 = \{\alpha_{01}, \alpha_{02}, \cdots, \alpha_{0n}\}.$$

Condition (6) and a theorem on implicit functions imply that the system (7b) defines continuously differentiable functions

$$y_i = g_i(x, A, B), \qquad (i = 1, 2, \cdots, n) \tag{9a}$$

in some neighborhood of the point (x_0, A_0, B_0). If we substitute these functions in the left members of equations (7a) we obtain continuously differentiable functions

$$v_i = h_i(x, A, B), \qquad (i = 1, 2, \cdots, n). \tag{9b}$$

We show that, in some neighborhood of the point x_0, the functions (9a) and (9b) satisfy the system (8) for arbitrary permissible values of the constant vectors A, B, that is, for values sufficiently close to A_0, B_0.

Now we must show that the constant vectors A, B can always be selected so that the curve (9a), (9b) passes through an arbitrary point taken sufficiently close to (x_0, Y_0, V_0). This follows from a theorem on implicit functions applied to the system (7a); this theorem guarantees that the system can be solved for $\beta_1, \beta_2, \cdots, \beta_n$ since condition (6) is satisfied.

In order to show that functions (9a), (9b) satisfy the system (8) we substitute the expressions (9a) for y_i, $(i = 1, 2, \cdots, n)$ in equations (7b) and differentiate the resulting identities with respect to x. This gives us

$$\frac{\partial^2 W}{\partial x \partial \beta_k} + \sum_{j=1}^{n} \frac{\partial^2 W}{\partial y_j \partial \beta_k} \frac{dy_j}{dx} = 0, \qquad (k = 1, 2, \cdots, n). \quad (10)$$

On the other hand if we substitute the expression for W in equation (5) and differentiate with respect to β_k, $(k = 1, 2, \cdots, n)$, we find that

$$\frac{\partial^2 W}{\partial x \partial \beta_k} = \sum_{j=1}^{n} \frac{\partial H}{\partial v_j} \frac{\partial^2 W}{\partial y_j \partial \beta_k}, \qquad (k = 1, 2, \cdots, n).$$

If we substitute for y_i, $(i = 1, 2, \cdots, n)$ from equation (9a) and compare the result with equation (10) we obtain the relations

$$\sum_{j=1}^{n} \frac{\partial^2 W}{\partial y_j \partial \beta_k} \left\{ \frac{dy_j}{dx} + \frac{\partial H}{\partial v_j} \right\} = 0, \qquad (k = 1, 2, \cdots, n).$$

Because, by equation (6), the determinant of the coefficients for the quantities

$$\frac{dy_j}{dx} + \frac{\partial H}{\partial v_j}, \qquad (j = 1, 2, \cdots, n)$$

is not zero, each of these quantities is zero. In other words the first of the canonical equations is satisfied.

Now consider equations (7a) which define v_i, $(i = 1, 2, \cdots, n)$; substitute the expression for y_i, $(i = 1, 2, \cdots, n)$ from equations (9a) in the left members of equations (7a). Differentiating the resulting equations with respect to x yields

$$\frac{dv_j}{dx} = \frac{\partial^2 W}{\partial y_j \partial x} + \sum_{k=1}^{n} \frac{\partial^2 W}{\partial y_j \partial y_k} \frac{dy_k}{dx} = \frac{\partial^2 W}{\partial y_j \partial x} - \sum_{k=1}^{n} \frac{\partial^2 W}{\partial y_j \partial y_k} \frac{\partial H}{\partial v_k}, \qquad (j = 1, 2, \cdots, n). \quad (10')$$

Now differentiate relation (5) with respect to y_j, $(j = 1, 2, \cdots, n)$

to obtain

$$\frac{\partial^2 W}{\partial x \partial y_j} = \frac{\partial H}{\partial y_j} + \sum_{k=1}^{n} \frac{\partial H}{\partial v_k} \frac{\partial^2 W}{\partial y_k \partial y_j}, \qquad (j = 1, 2, \cdots, n).$$

Using this and equation (10′) we find that

$$\frac{dv_j}{dx} = \frac{\partial H}{\partial y_j}, \qquad (j = 1, 2, \cdots, n).$$

This completes the proof of Theorem 2.

In the concluding paragraphs of this section we consider some geometric notions connected with a field. Here the most obvious and also the most typical problems of the calculus of variations are the problems of finding geodesic lines. This suggests a geometric interpretation or at least the use of geometric language for arbitrary calculus of variations problems. For this purpose we consider a special "metric" in which the "length" of a curve $Y = Y(x)$, $(a \leq x \leq b)$ is the value of the functional

$$J[Y] = \int_a^b f(x, Y, Y')\, dx.$$

The extremals of this functional are "geodesics" and the "distance" between two points is the "length" of the extremal joining these points.

Consider a Mayer family of extremals of our functional $J[Y]$, for example, the collection of all extremals passing through some fixed point. This family generates a field and a function $W(x, Y)$, the Hilbert integral. Consider the family of hypersurfaces

$$W(x, Y) = C = \text{const.} \tag{11}$$

For $n = 2$, this is a surface and, for $n = 1$, it is a curve. Now take two hypersurfaces corresponding to $C = C_1$ and $C = C_2 > C_1$. We show that the segment of an arbitrary "geodesic" included between these hypersurfaces has "length" $C_2 - C_1$. The considered "length" is equal to the integral

$$J[\bar{Y}] = \int_{(x_1, Y_1)}^{(x_2, Y_2)} f(x, \bar{Y}, \bar{Y}')\, dx$$

taken along the designated arc of the extremal whose end points are (x_1, Y_1), (x_2, Y_2). Along an extremal (trajectory) of the field the fundamental functional is equal to the Hilbert integral. Hence the "length" with which we are concerned is equal to

$$\int_{(x_1,Y_1)}^{(x_2,Y_2)} \left[f - \sum_{i=1}^{n} p_i(x, Y) f_{y_i'} \right] dx + \sum_{i=1}^{n} f_{y_i'} dy_i = W(x, Y_2) - W(x, Y_1) = C_2 - C_1.$$

This proves our assertion.

We see that these hypersurfaces are "equidistant." Hence it is natural to ask whether, at the point of intersection of an extremal of the field and the hypersurface (11), there is a definite dependence between the direction of the extremal and the direction of an arbitrary curve passing through this point and lying on the hypersurface. Such a dependence must play the role of orthogonality in the illustration that we are considering. Let an element of the curve have components

$$\delta x, \delta y_1, \delta y_2, \cdots, \delta y_n.$$

If it is an extremal, its slope at the considered point (x, Y) is defined by the formulas

$$\frac{dy_i}{dx} = p_i(x, Y), \qquad (i = 1, 2, \cdots, n).$$

The function $W(x, Y)$ has a constant value on the hypersurface (11); thus, at the point under consideration,

$$\frac{\partial W}{\partial x} \delta x + \sum_{i=1}^{n} \frac{\partial W}{\partial y_i} \delta y_i = 0$$

or

$$\left[f\left(x, Y, \frac{dY}{dx}\right) - \sum_{i=1}^{n} \frac{dy_i}{dx} f_{y_i'}\left(x, Y, \frac{dY}{dx}\right) \right] \delta x + \sum_{i=1}^{n} f_{y_i'}\left(x, Y, \frac{dY}{dx}\right) \delta y_i = 0. \quad (12)$$

We have obtained a completely defined relation between the direction of an element of a curve on the hypersurface and the slope of the extremal of the field. This relation is linear relative to the components of an element of a curve on the hypersurface. With respect to the components of the slope of the extremal, it is in general not linear.

For the case in which

$$f(x, Y, Z) = n(x, Y)\sqrt{1 + z_1^2 + z_2^2 + \cdots + z_n^2}$$

relation (12) becomes

$$\delta x + \sum_{i=1}^{n} \frac{dy_i}{dx}\,\delta y_i = 0,$$

that is, it becomes the orthogonality relation. The reader can easily verify this. In the general case, relation (12) is called the *transversality relation* and the hypersurface (11) is said to be *transversal* to the given field of extremals.

For the reader who is interested in the development of geometric notions touched upon here we recommend a book by M. A. Lavrent'ev and L. A. Lyusternik, *Osnovy variacionnogo iscisleniya* (Foundations of the Calculus of Variations), Vol. 1, Part 2, Moscow and Leningrad, 1935.

EXERCISES

1. Show that

$$y = -\tfrac{1}{4}x^2 + 1$$

furnishes a strong relative minimum for

$$\int_0^2 (xy' + y'^2)\,dx, \qquad y(0) = 1, y(2) = 0.$$

2. Show that $y = 0$ furnishes a strong relative minimum for

$$\int_0^a (y'^2 + 2yy' - 16y^2)\,dx; \qquad a > 0, y(0) = 0, y(a) = 0$$

if $0 < a < \frac{1}{4}\pi$ but that there is no minimum if $a > \frac{1}{4}\pi$.

3. Show that no continuous function furnishes an extremum for

$$\int_{-1}^{2} y'(1 + x^2y')\,dx; \qquad y(-1) = 1, y(2) = 4.$$

4. Show that

$$y = 7 - \frac{4}{x}$$

furnishes a strong relative minimum for

$$\int_{1}^{2} y'(1 + x^2y')\,dx; \qquad y(1) = 3, y(2) = 5.$$

5. Show that $y = \sin 2x - 1$ furnishes a strong relative maximum for

$$\int_{0}^{\frac{1}{4}\pi} (4y^2 - y'^2 + 8y)\,dx; \qquad y(0) = -1, y(\tfrac{1}{4}\pi) = 0.$$

6. Show that $y = x^2$ furnishes a weak relative minimum for

$$\int_{1}^{2} \frac{x^3}{y'^2}\,dx; \qquad y(1) = 1, y(2) = 4.$$

7. Show that $y = bx/a$ furnishes a weak relative minimum but not a strong relative minimum for

$$\int_{0}^{a} y'^3\,dx; \qquad y(0) = 0, y(a) = b, a > 0, b > 0.$$

3

Generalizations of the Fundamental Problem

20. VARIATIONS OF A FUNCTIONAL

To derive the first necessary condition for an extremum, we constructed a one-parameter family $Y(x, \epsilon)$ of admissible vector functions containing the vector function $\bar{Y}(x)$ that furnished an extremum for the functional we were studying. That is, we imbedded the vector function $\bar{Y}(x)$ in a one-parameter family of admissible vector functions. Because we were discussing an extremum in a class of vector functions that assumed given values at the end points of an interval we set

$$Y(x, \epsilon) = \bar{Y}(x) + \epsilon H(x)$$

in which $H(x)$ was an arbitrary piecewise-smooth vector function equal to zero at the end points of the interval.

More complicated boundary conditions are often imposed. The end points of admissible curves may lie on given curves or surfaces rather than being fixed; and additional relations or conditions may be imposed on the desired function. In such a case the problem of imbedding an admissible function in a family of similar functions presents some difficulty. To solve this problem we consider a series of very important generalizations of the fundamental problem. Here, however, we suppose that the family has already been constructed.

For simplicity we suppose that $n = 1$, that is, we consider curves in a plane. The family of admissible functions is given by the equation

$$y = \omega(x, \epsilon). \tag{1}$$

Here $\omega(x, \epsilon)$ and $\omega_\epsilon(x, \epsilon)$ are piecewise-smooth functions of x, and

$$\omega(x, 0) = \bar{y}(x).$$

Thus the curves (1) vary with changes in the value of the parameter ϵ; although they are deformed they continue to satisfy boundary conditions and additional restrictions. In general an explicit expression for the function $\omega(x, \epsilon)$ is not necessary; the possibility of constructing the family in principle is all that is necessary. It is however often essential for the construction of the family that a definite arbitrariness remain; in the fundamental problem this was a consequence of the arbitrary choice of the piecewise-smooth vector function $H(x)$. At this stage of the investigation, the arbitrariness is not important; thus we may simply express our functional $J[y]$ in terms of the family (1), and ignore the restrictions and boundary conditions because they are automatically satisfied. The functional $J[y]$ is a function of ϵ and if the function $\bar{y}(x)$ gives this functional an extremum, the derivative of $J[y]$ with respect to ϵ is equal to zero for $\epsilon = 0$:

$$\frac{d}{d\epsilon} J[y]|_{\epsilon=0} = 0.$$

We can take the differential as well as the derivative with respect to ϵ; it is designated by the symbol δ instead of d to distinguish it from the differential with respect to x. This differential $\delta J[y]$ is called the *first variation of the functional.* Thus, in the simplest case of the fundamental problem in which

$$J[y] = \int_a^b f(x, y, y')\, dx, \qquad [y(a) = a_1, y(b) = b_1]$$

and

$$\omega(x, \epsilon) = \bar{y}(x) + \epsilon\eta(x)$$

the first variation is equal to*

$$\delta\bar{J} = \frac{d}{d\epsilon} J[y]\bigg|_{\epsilon=0} \cdot \epsilon = \epsilon \int_a^b \{\bar{f}_y \eta + \bar{f}_{y'}\eta'\}\, dx.$$

The computation of the first variation in the case of the funda-

*Because we are considering the neighborhood of the point $\epsilon = 0$, we write ϵ instead of $\delta\epsilon$.

mental problem led us to the first necessary condition for an extremum, the Euler equation. This equation can be considered as necessary and sufficient that the first variation of the functional $J[y]$ vanish; that is, the functional is *stationary* if and only if the Euler equation is satisfied. In the case of the fundamental problem we have considered *the second variation*

$$\delta^2 \bar{J} = \frac{d^2}{d\epsilon^2} J[y]\bigg|_{\epsilon=0} \cdot \epsilon^2 = \epsilon^2 \int_a^b \{\bar{f}_{yy}\eta^2 + 2\bar{f}_{yy'}\eta\eta' + \bar{f}_{y'y'}\eta'^2\}\, dx$$

which is a differential of the second order in ϵ. The functional $I_2[u]$ introduced in Section 18 differs from the second variation only by a constant multiplier.

As differentials of various orders appear in the expansion of a finite increment of a function, the variations of a functional appear with proper multipliers as successive terms in the expression for the finite increment of the functional:

$$\Delta J = J[y] - J[\bar{y}] = \delta \bar{J} + \frac{1}{2!}\delta^2 \bar{J} + \frac{1}{3!}\delta^3 \bar{J} + \cdots .$$

The name "calculus of variations" originated in the notions presented in this section. In applications, the calculus of variations most often serves to derive differential equations which certain functions must satisfy or relations which these functions must satisfy at boundary points or corner points. Only the first variation is needed to derive these. Thus we shall study only the first variation in this chapter and use it to analyze various generalizations of the fundamental problem. Our analysis does not permit a summary of necessary or sufficient conditions for minima or maxima. To do this we should generalize in some measure the results of the second chapter; such generalizations exceed the scope of this book.

21. VARIABLE END POINTS

Suppose we wish to find an extremum for a functional in a class of curves with one end point fixed and the other end point on some geometric locus (curve, surface, or hypersurface). Because

these curves include those curves with both end points fixed, the desired curve must be an ordinary (smooth or broken) extremal. In other words, the equations that define the desired curve are the Euler equations of Section 3, Chapter 1. But the extremal we are seeking must also satisfy the condition that defines the variable end point.

We restrict ourselves to the case $n = 1$ and take the functional in parametric form,

$$J_C = \int_C F(x, y, \dot{x}, \dot{y})\, dt.$$

Suppose that the terminal point M_2 of the family is fixed and the initial point M_1 lies on some curve L with parametric equations

$$L: \qquad x = \phi(\tau), \qquad y = \psi(\tau), \qquad (\tau_1 \leq \tau \leq \tau_2).$$

We assume that $\phi(\tau), \psi(\tau)$ have continuous derivatives $\phi'(\tau), \psi'(\tau)$ that do not vanish simultaneously.

Let us suppose that the functional J_C has an extremum in the class of admissible curves we are considering and that it attains this extremum for the smooth* extremal $\bar{C}$. We denote the value of the parameter τ at the point of intersection of the curve L with the extremal $\bar{C}$ by $\bar{\tau}$ and we assume that $\tau_1 < \bar{\tau} < \tau_2$; if L is a simple arc we may assume that the point of intersection is different from the end points.

Suppose the equation of the extremal $\bar{C}$ has the form

$$x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad (t_1 \leq t \leq t_2)$$

so that

$$\bar{x}(t_1) = \phi(\bar{\tau}), \qquad \bar{y}(t_1) = \psi(\bar{\tau}).$$

Our assumptions about the curve L insure that at least one of the quantities $\phi'(\bar{\tau}), \psi'(\bar{\tau})$ is different from zero; suppose that

$$\phi'(\bar{\tau}) \neq 0.$$

Now we vary the extremal $\bar{C}$ by introducing the curve with equations

$$\begin{aligned} x &= \bar{x}(t) + \epsilon_1(t - t_2), \\ y &= \bar{y}(t) + \epsilon_2(t - t_2), \qquad (t_1 \leq t \leq t_2). \end{aligned} \tag{1}$$

*The terminology introduced in Section 4 allows us to omit the word smooth.

This curve passes through the point M_2 for all values of the parameters ϵ_1, ϵ_2. The condition that the variable end point M_1 must satisfy leads to two equations

$$\bar{x}(t_1) + \epsilon_1(t_1 - t_2) = \phi(\tau),$$
$$\bar{y}(t_1) + \epsilon_2(t_1 - t_2) = \psi(\tau).$$

In some neighborhood of $\epsilon_1 = 0$, the first of these equations defines a continuously differentiable function $\tau = \tau(\epsilon_1)$ that coincides with $\bar{\tau}$ for $\epsilon_1 = 0$. If we substitute this function for τ in the second equation we find that ϵ_2 is a continuously differentiable function of ϵ_1; it approaches zero with ϵ_1 and we have

$$\left(\frac{d\epsilon_2}{d\epsilon_1}\right)_{\epsilon_1=0} = \frac{\psi'(\bar{\tau})}{\phi'(\bar{\tau})}.$$

On a curve C of the family (1) the functional J_C is a function of the parameter ϵ_1:

$$J_C = \int_{t_1}^{t_2} F(\bar{x} + \epsilon_1(t - t_2), \bar{y} + \epsilon_2(t - t_2), \bar{x}' + \epsilon_1, \bar{y}' + \epsilon_2)\, dt = \Phi(\epsilon_1).$$

This function has an extremum for $\epsilon_1 = 0$. Consequently

$$\Phi'(\epsilon_1) = 0$$

which implies that the first variation δJ_C of the functional J_C is zero along the curve $\bar{C}$. Let us integrate by parts:

$$\delta J_C = \delta\epsilon_1 \int_{t_1}^{t_2} \left\{F_x(t - t_2) + F_y(t - t_2)\frac{d\epsilon_2}{d\epsilon_1} + F_{\dot{x}} + F_{\dot{y}}\frac{d\epsilon_2}{d\epsilon_1}\right\} dt$$

$$= -\delta\epsilon_1(t_1 - t_2)\left\{F_{\dot{x}} + \frac{d\epsilon_2}{d\epsilon_1}F_{\dot{y}}\right\}\Bigg|_{t=t_1}$$

$$+ \delta\epsilon_1 \int_{t_1}^{t_2} (t - t_2)\left\{\left(F_x - \frac{d}{dt}F_{\dot{x}}\right) + \frac{d\epsilon_2}{d\epsilon_1}\left(F_y - \frac{d}{dt}F_{\dot{y}}\right)\right\} dt.$$

If we set $\epsilon_1 = 0$ and recall that $\bar{x}(t), \bar{y}(t)$ satisfy the Euler equa-

tion we find

$$\delta J_{\bar{C}} = -\delta\epsilon_1(t_1 - t_2)\left\{\bar{F}_{\dot{x}} + \left(\frac{d\epsilon_2}{d\epsilon_1}\right)_{\epsilon_1=0}\bar{F}_{\dot{y}}\right\}\Bigg|_{t=t_1}$$

$$= -\delta\epsilon_1(t_1 - t_2)\left\{\bar{F}_{\dot{x}} + \frac{\psi'(\bar{\tau})}{\phi'(\bar{\tau})}\bar{F}_{\dot{y}}\right\}\Bigg|_{t=t_1}.$$

This quantity must be zero. Thus we obtain the relation

$$\bar{F}_{\dot{x}}\phi'(\bar{\tau}) + \bar{F}_{\dot{y}}\psi'(\bar{\tau})|_{t=t_1} = 0.$$

It is convenient to express this in another way. We designate the coordinates of the initial point of the curve $\bar{C}$ by $\bar{x}_1, \bar{y}_1$, their differentials along $\bar{C}$ by $d\bar{x}_1, d\bar{y}_1$ and along L by $\delta\bar{x}_1, \delta\bar{y}_1$. If we notice that

$$\delta\bar{x}_1 = \delta\epsilon_1(t_1 - t_2)$$

$$\delta\bar{y}_1 = \delta\epsilon_1(t_1 - t_2)\left(\frac{d\epsilon_2}{d\epsilon_1}\right)_{\epsilon_1=0}$$

we may write

$$\delta J_{\bar{C}} = -\{F_{\dot{x}}(\bar{x}_1, \bar{y}_1, d\bar{x}_1, d\bar{y}_1)\,\delta\bar{x}_1 + F_{\dot{y}}(\bar{x}_1, \bar{y}_1, d\bar{x}_1, d\bar{y}_1)\,\delta\bar{y}_1\}.$$

The equation $\delta J_{\bar{C}} = 0$ then becomes

$$F_{\dot{x}}(\bar{x}_1, \bar{y}_1, d\bar{x}_1, d\bar{y}_1)\,\delta\bar{x}_1 + F_{\dot{y}}(\bar{x}_1, \bar{y}_1, d\bar{x}_1, d\bar{y}_1)\,\delta\bar{y}_1 = 0.$$

This expresses the relation between the slope of the extremal and the curve L at the variable end point. If we adjoin the equation

$$F_{\dot{x}}(x, y, dx, dy)\,\delta x + F_{\dot{y}}(x, y, dx, dy)\,\delta y = 0 \qquad (2)$$

to the equations of the pencil of extremals with vertex at M_2 and to the equations of the curve L we arrive at a system of equations for the coordinates of the variable end point.

Relation (2) is a simple case of the transversality relation (or condition) in Section 19; thus the coordinates of the variable end point are subject to the transversality condition. If we apply it to the present problem, we can write the transversality condition of Section 19 in the form:

$$[F(x, y, dx, dy) - F_{\dot{x}}(x, y, dx, dy)\,dx - F_{\dot{y}}(x, y, dx, dy)\,dy]\,\delta t + F_{\dot{x}}(x, y, dx, dy)\,\delta x + F_{\dot{y}}(x, y, dx, dy)\,\delta y = 0.$$

Because of the homogeneity of F, the coefficient of δt must be zero.

We can treat the functional in ordinary form

$$J[y] = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

in much the same way. If we suppose that the left-hand end point (x_1, y_1) lies on a given curve L, the relation at the left-hand end point can be written in the form

$$[f(x_1, y_1, y_1') - y_1' f_{y'}(x_1, y_1, y_1')]\, \delta x_1 + f_{y'}(x_1, y_1, y_1')\, \delta y_1 = 0 \quad (3)$$

which is identical with relation (12), Section 19, for $n = 1$.

We can obtain this result formally by reducing the functional $J[y]$ to the functional in parametric form

$$\int_{t_1}^{t_2} F(x, y, \dot{x}, \dot{y})\, dt$$

by setting

$$F(x, y, \dot{x}, \dot{y}) = f\left(x, y, \frac{\dot{y}}{\dot{x}}\right) \dot{x}.$$

Since the relations

$$F_{\dot{x}} = f - \frac{\dot{y}}{\dot{x}} f_{y'} = f - y' f_{y'},$$

$$F_{\dot{y}} = f_{y'}$$

hold, we may write

$$F_{\dot{x}}\, \delta x + F_{\dot{y}}\, \delta y = (f - y' f_{y'})\, \delta x + f_{y'}\, \delta y.$$

The problem in which no condition is imposed at the end point of the interval of integration is a simple case of a variable end point problem. This is called the problem with *natural condition* (at the end point considered). If the left end point is unrestricted and the right end point is fixed, the problem is formulated in the the following way: Find an extremum for the functional

$$J[y] = \int_a^b f(x, y, y')\, dx$$

with the condition $y(b) = b_1$.

This problem is included in our general problem if the left-hand end point must lie on the straight line

$$x = a.$$

Here the straight line is the curve L. Then $\delta x = 0$ for the left-hand end point; this means that the condition for the left-hand end point has the form

$$f_{y'}(x, y, y')|_{x=a} = 0.$$

Sometimes it is necessary to consider the variable end point problem (again we take the left) for a more general functional

$$J[y] = \int_{x_1}^{x_2} f(x, y, y')\, dx - g(x_1, y_1).$$

Here $g(x_1, y_1)$ is a given continuously differentiable function of the coordinates of the left-hand end point which is to lie on the curve L; the right-hand end point is assumed to be fixed. Instead of condition (3) we arrive at the condition

$$[f(x_1, y_1, y_1') - y_1' f_{y'}(x_1, y_1, y_1')]\, \delta x_1 + f_{y'}(x_1, y_1, y_1')\, \delta y_1 + g_x(x_1, y_1)\, \delta x_1 + g_y(x_1, y_1)\, \delta y_1 = 0. \quad (4)$$

We have supposed the left-hand end point to be variable and the right fixed. The method used to solve the variable end point problem is the same whichever end point is variable and also for both end points variable. In the latter case we obtain two equations, one for each end point. The derivation of these presents no difficulty.

To conclude this section we consider a so-called *discontinuous problem* of the calculus of variations:

Find an extremum for the functional

$$J[y] = \int_a^b f(x, y, y')\, dx$$

in which $f(x, y, z)$ is not continuous as a function of $(x, y) \in G$. In this problem, a curve L that has a continuous tangent divides the region G into two parts G_1 and G_2. In each of these parts $f(x, y, z)$ satisfies all the conditions laid down in Section 1 for arbitrary finite z. On the curve L, however, the continuity of the

function $f(x, y, z)$ breaks down; in other words, this curve is a curve of discontinuity of the first kind for the function $f(x, y, z)$. The left-hand end point (a, a_1) lies in G_1 and the right-hand end point (b, b_1) lies in G_2. Under these conditions an extremum is sought for the functional $J[y]$.

If we assume that the desired curve intersects the curve L once (at a point with coordinates $\bar{x}, \bar{y}$) it is natural to set

$$J[y] = J_1[y] + J_2[y] = \int_a^{\bar{x}} f(x, y, y')\,dx + \int_{\bar{x}}^b f(x, y, y')\,dx$$

in which

$$f = \begin{cases} f_1 & \text{if } (x, y) \in G_1 \\ f_2 & \text{if } (x, y) \in G_2. \end{cases}$$

The desired curve is a broken curve with its corner point on the curve L; one of its links is an extremal for $J_1[y]$ and the other an extremal for $J_2[y]$. We assume that each of these links is a smooth extremal. The corner point is a variable point for each, and in finding the variation of the functional by the method presented above it is possible to obtain the condition defining this corner point. The condition has the form

$$\begin{aligned}\delta\bar{x}[f_1(\bar{x}, \bar{y}, \bar{y}_1') - \bar{y}_1' f_{1y'}(\bar{x}, \bar{y}, \bar{y}_1')] + \delta\bar{y} f_{1y'}(\bar{x}, \bar{y}, \bar{y}_1') \\ = \delta\bar{x}[f_2(\bar{x}, \bar{y}, \bar{y}_2') - \bar{y}_2' f_{2y'}(\bar{x}, \bar{y}, \bar{y}_2')] + \delta\bar{y} f_{2y'}(\bar{x}, \bar{y}, \bar{y}_2').\end{aligned}$$

This equation is a generalization of the Weierstrass-Erdmann condition. Suppose that the function $f(x, y, z)$ is continuous. Then, since the quantities $\delta\bar{x}, \delta\bar{y}$ are arbitrary, first set $\delta\bar{x} = 0$, $\delta\bar{y} \neq 0$ and then set $\delta\bar{x} \neq 0, \delta\bar{y} = 0$ to obtain the Weierstrass-Erdmann corner condition for the corner point $(\bar{x}, \bar{y})$.

22. VARIATIONS OF DOUBLE INTEGRALS

Consider the functional

$$J[u] = \iint_D f(x, y, u, u_x, u_y)\,dx\,dy \tag{1}$$

in which the function $f(x, y, u, p, q)$ has continuous second-order

derivatives in all its arguments for arbitrary finite p, q when (x, y, u) belongs to a region G of the given space. In this region G we have a continuous closed space curve L, distinct points of which have distinct projections on the xy-plane. Suppose the projection of L on the xy-plane is a piecewise-smooth curve C that bounds a region D. Consider the collection of all piecewise-smooth surfaces

$$u = u(x, y)$$

that lie in G and contain L; we call such surfaces *admissible.** Suppose we know that there exist† admissible surfaces on which $J[u]$ has finite values. The problem is to find surfaces on which the functional has extreme values. This is the problem of finding absolute extrema, but we can easily define a weak or strong neighborhood of a given surface and state the problem of finding relative extrema.

Here we restrict ourselves to the problem of finding a first necessary condition, that is, the problem of finding the equation of a surface that may furnish an extremum.

Suppose that an admissible surface

$$u = \bar{u}(x, y) \tag{2}$$

gives the functional (1) a weak extremum and that the function $\bar{u}(x, y)$ has continuous derivatives of the second order. We shall show that $\bar{u}(x, y)$ satisfies the Euler equation

$$f_u - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q = 0. \tag{3}$$

This is the Euler equation for the functional (1); each of its integral surfaces is called an extremal.

Contrast this development to our development for the simplest problem; there, we first obtained the Euler equation in integral form and then demonstrated that $\bar{y}(x)$ has a second derivative at each nonsingular extremal. Development of the theory for double-

*In other words, consider the collection of all piecewise-smooth functions $u(x, y)$ in D which become a given continuous function on the boundary C of this region.

†See the Appendix, Section A-23.

integral functionals without the assumption that $\bar{u}(x, y)$ has continuous second-order derivatives is complicated but not impossible; it has not been carried out as far as that for the simplest problem.*

To prove that if $\bar{u}(x, y)$ furnishes an extremum it satisfies equation (3), we find the first variation of the functional (1) and set it equal to zero. To imbed the surface (2) in a one-parameter family of admissible surfaces, it is sufficient to set

$$u = \bar{u}(x, y) + \epsilon\zeta(x, y)$$

in which $\zeta(x, y)$ is an arbitrary smooth or piecewise-smooth function that vanishes on the curve C. The first variation of the functional is equal to

$$\delta\bar{J} = \iint_D \{\bar{f}_u\zeta + \bar{f}_p\zeta_x + \bar{f}_q\zeta_y\}\, dx\, dy.$$

We notice that

$$\iint_D \{\bar{f}_p\zeta_x + \bar{f}_q\zeta_y\}\, dx\, dy = \iint_D \left\{\frac{\partial}{\partial x}(\zeta\bar{f}_p) + \frac{\partial}{\partial y}(\zeta\bar{f}_q)\right\} dx\, dy - \iint_D \zeta\left\{\frac{\partial}{\partial x}\bar{f}_p + \frac{\partial}{\partial y}\bar{f}_q\right\} dx\, dy$$

and also that

$$\iint_D \left\{\frac{\partial}{\partial x}(\zeta\bar{f}_p) + \frac{\partial}{\partial y}(\zeta\bar{f}_q)\right\} dx\, dy = \int_C \zeta\{\bar{f}_p\, dy - \bar{f}_q\, dx\}.$$

Since $\zeta = 0$ on C, we find $\delta\bar{J}$:

$$\delta\bar{J} = \iint_D \zeta\left\{\bar{f}_u - \frac{\partial}{\partial x}\bar{f}_p - \frac{\partial}{\partial y}\bar{f}_q\right\} dx\, dy.$$

Thus if a surface furnishes an extremum, it satisfies the relation

$$\iint_D \zeta\left\{\bar{f}_u - \frac{\partial}{\partial x}\bar{f}_p - \frac{\partial}{\partial y}\bar{f}_q\right\} dx\, dy = 0. \tag{4}$$

*See the Appendix, Section A-29.

Here ζ is the arbitrary function mentioned above that vanishes on C. Now we must prove that at every point of the region D the multiplier of ζ under the integral sign vanishes.* To prove this, we assume that the quantity

$$M(x, y) = \bar{f}_u - \frac{\partial}{\partial x}\bar{f}_p - \frac{\partial}{\partial y}\bar{f}_q$$

is not zero at some interior point of D; suppose it is positive. Continuity insures that $M(x, y)$ is positive in a circle of sufficiently small radius with center (x_0, y_0). We set

$$\bar{\zeta}(x, y) = \begin{cases} 0 & \text{for } (x-x_0)^2+(y-y_0)^2 \geq \rho^2 \\ [\rho^2-(x-x_0)^2-(y-y_0)^2]^\nu > 0 & \text{for } (x-x_0)^2+(y-y_0)^2 < \rho^2 \end{cases}$$

where $\nu > 1$. This function is zero on C and m times continuously differentiable in the entire plane if $\nu > m$. Equation (4) implies that

$$\iint_D \bar{\zeta}(x, y)M(x, y)\,dx\,dy = 0$$

but

$$\iint_D \bar{\zeta}(x, y)M(x, y)\,dx\,dy = \iint_{(x-x_0)^2+(y-y_0)^2<\rho^2} \bar{\zeta}(x, y)M(x, y)\,dx\,dy > 0.$$

This contradiction proves our assertion.

Differential equation (3) goes back to Lagrange [1760–1761, *Miscellanea Taurinensia*, Vol. 2 (Reale Accademia delle Scienze di Torino, Miscellanea-Philosophico-Mathematica Societatis privatae Taurinensis)] at least for simple cases. He stated the prob-

*This result still holds if the class of functions to which $\zeta(x, y)$ belongs is the smaller class of functions that vanish on C and are m times continuously differentiable for any fixed number m.

We note that we have here the two-dimensional analogue of Lagrange's lemma, proved in Section 5. At the end of this section we formulate the corresponding lemma for the n-dimensional case.

lem for *minimal surfaces*. In this problem, the functional is the area of part of a surface bounded by L:

$$J[u] = \iint_D \sqrt{1 + p^2 + q^2}\, dx\, dy, \qquad \left[p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y}\right]$$

if the ordinary form of the equation of a surface is used. Using equation (3), we write the equation that defines the minimal surfaces in the form

$$\frac{\partial}{\partial x}\frac{p}{\sqrt{1 + p^2 + q^2}} + \frac{\partial}{\partial y}\frac{q}{\sqrt{1 + p^2 + q^2}} = 0. \tag{5}$$

Lagrange remarked that this equation states the fact that the quantity

$$\frac{p\, dy - q\, dx}{\sqrt{1 + p^2 + q^2}}$$

is an exact differential.*

Equation (5) can be written in the form

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

in which

$$r = \frac{\partial^2 u}{\partial x^2}, \qquad s = \frac{\partial^2 u}{\partial x \partial y}, \qquad t = \frac{\partial^2 u}{\partial y^2}.$$

The geometric significance of the equation in p, q, r, s, and t is this: at each point of the surface the mean curvature is zero. This was first discovered by Meusnier in 1776.

A general expression for the variation of an integral of arbitrary multiplicity was first obtained by M. V. Ostrogradski in 1834 both in the case of a fixed boundary and in the general case of a variable boundary.†

The results of this section can be extended to the case of a triple integral and a fixed boundary without difficulty; this is recom-

*In the Appendix, Section A-29, we shall return to this remark by Lagrange.

†The memoir of Ostrogradski was first published in the writings of the Petersburg Academy of Science. It is entitled "Mémoire sur le calcul des variations des intégrales multiples." It was reprinted in the 15th volume of *Journal für die reine und angewandte Mathematik*, A. L. Crelle, ed., 1836.

mended to the reader as an exercise. For this purpose we need the special case for $n = 3$ of the following lemma:

If $M(x_1, x_2, \cdots, x_n)$ is a continuous function in some bounded n-dimensional region G and if

$$\iint_G \cdots \int M(x_1, x_2, \cdots, x_n)\zeta(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n = 0$$

for every function ζ that is zero on the boundary of G and is continuously differentiable m times in the closure of G, where m is some fixed number, then

$$M(x_1, x_2, \cdots, x_n) = 0.$$

This lemma is proved for arbitrary n in exactly the same way it was proved above for $n = 2$.

23. VARIATIONS OF FUNCTIONALS THAT DEPEND ON DERIVATIVES OF HIGHER ORDER

We consider first the single integral

$$J[y] = \int_a^b f(x, y, y', y'', \cdots, y^{(n)})\, dx \tag{1}$$

in the class of functions $y(x)$ having continuous derivatives up to and including order $n - 1$ and having a piecewise-continuous derivative of order n. The curve $y = y(x)$ must lie in some region G of the xy-plane and must satisfy given boundary conditions

$$y(a) = a_0, \qquad y'(a) = a_1, \cdots, y^{(n-1)}(a) = a_{n-1}$$
$$y(b) = b_0, \qquad y'(b) = b_1, \cdots, y^{(n-1)}(b) = b_{n-1}.$$

The class of all such admissible curves is denoted by $\mathfrak{M}$. We assume that the function $f(x, y, z_1, z_2, \cdots, z_n)$ has continuous derivatives up to and including order $n + 1$ in all its arguments for $(x, y) \in G$ and for arbitrary finite values of $z_1, z_2, \cdots, z_n$.

We wish to find an extremum for the functional in the class $\mathfrak{M}$; that is, we wish to find an absolute extremum. We shall introduce neighborhoods of various orders from zero to n and state the problem for relative extrema of various orders; then we shall single out two cases: weak (nth order) and strong (zero order).

We want to determine the fundamental condition an admissible curve $y = \bar{y}(x)$ lying in G (except possibly for its end points) must satisfy if it gives the functional an extremum. We obtain this condition by setting the first variation of the functional $J[y]$ equal to zero. We imbed the curve $y = \bar{y}(x)$ in a one-parameter family and set

$$y = \bar{y}(x) + \epsilon\eta(x)$$

in which the function $\eta(x)$ has continuous derivatives up to and including that of order $n - 1$ and a piecewise-continuous nth order derivative such that

$$\eta(a) = \eta'(a) = \cdots = \eta^{(n-1)}(a) = 0 \tag{2a}$$

$$\eta(b) = \eta'(b) = \cdots = \eta^{(n-1)}(b) = 0. \tag{2b}$$

First of all we have:

$$\delta\bar{J} = \epsilon\int_a^b \{\bar{f}_y\eta + \bar{f}_{y'}\eta' + \bar{f}_{y''}\eta'' + \cdots + \bar{f}_{y^{(n)}}\eta^{(n)}\}\, dx.$$

Integrating by parts and using the boundary conditions (2a) and (2b), we find that

$$\delta\bar{J} = \epsilon\int_a^b \left\{\left[\bar{f}_{y'} - \int_a^x \bar{f}_y\, dx\right]\eta' + \bar{f}_{y''}\eta'' + \cdots + \bar{f}_{y^{(n)}}\eta^{(n)}\right\} dx.$$

Further integration by parts gives

$$\delta\bar{J} = \epsilon\int_a^b \left\{\left[\bar{f}_{y''} - \int_a^x \left(\bar{f}_{y'} - \int_a^x \bar{f}_y\, dx\right) dx\right]\eta'' + \cdots + \bar{f}_{y^{(n)}}\eta^{(n)}\right\} dx.$$

Continuing this process we obtain:

$$\delta\bar{J} = \epsilon\int_a^b M_n(x)\eta^{(n)}(x)\, dx \tag{3}$$

in which

$$M_n(x) = \bar{f}_{y^{(n)}} - \int_a^x M_{n-1}(x)\, dx$$

$$M_{n-1}(x) = \bar{f}_{y^{(n-1)}} - \int_a^x M_{n-2}(x)\, dx$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$M_1(x) = \bar{f}_{y'} - \int_a^x \bar{f}_y\, dx.$$

The first variation (3) must be zero for arbitrary functions $\eta(x)$ that satisfy the conditions prescribed above. We show* that

$$M_n(x) = C_0 + C_1(x - a) + C_2(x - a)^2 + \cdots + C_{n-1}(x - a)^{n-1}$$

where $C_0, C_1, \cdots, C_{n-1}$ are constants.

Indeed, we define the constants $C_0, C_1, \cdots, C_{n-1}$ so that

$$\int_a^b \{M_n(x) - C_0 - C_1(x - a) - C_2(x - a)^2 - \cdots - C_{n-1}(x - a)^{n-1}\}(x - a)^k\, dx = 0 \quad (4)$$

for $k = 0, 1, 2, \cdots, n - 1$. This is possible because the system of equations (4) can be expressed in the form

$$C_0 \frac{(b - a)^{k+1}}{k + 1} + C_1 \frac{(b - a)^{k+2}}{k + 2} + \cdots + C_{n-1} \frac{(b - a)^{k+n}}{k + n} = \int_a^b M_n(x)(x - a)^k\, dx, \qquad (k = 0, 1, 2, \cdots, n - 1)$$

and its determinant is different from zero:

$$\begin{vmatrix} \frac{b-a}{1} & \frac{(b-a)^2}{2} & \cdots & \frac{(b-a)^n}{n} \\ \frac{(b-a)^2}{2} & \frac{(b-a)^3}{3} & \cdots & \frac{(b-a)^{n+1}}{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{(b-a)^n}{n} & \frac{(b-a)^{n+1}}{n+1} & \cdots & \frac{(b-a)^{2n-1}}{2n-1} \end{vmatrix}$$

*The assertion that the piecewise-continuous function $M(x)$, $(a \leq x \leq b)$ is equal to a polynomial of degree $n - 1$ if

$$\int_a^b M(x)\eta^{(n)}(x)\, dx = 0$$

for every function $\eta(x)$ that has a piecewise-continuous derivative of order n and satisfies conditions (2a) and (2b) is often called the *second lemma of the calculus of variations*. For the case $n = 1$, it is called the *Du Bois Reymond lemma;* we proved this lemma in Section 3. An essential generalization of the second lemma is given in Section 24 of the Appendix.

$$= (b-a)^{n^2} \begin{vmatrix} \frac{1}{1} & \frac{1}{2} & & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & & \frac{1}{2n-1} \end{vmatrix}$$

$$= (b-a)^{n^2} \frac{[1!2! \cdots (n-1)!]^3}{n!(n+1)! \cdots (2n-1)!}.$$

Having found the constants $C_0, C_1, \cdots, C_{n-1}$, we set

$$\tilde{\eta}(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \{M_n(t) - C_0 - C_1(t-a) - \cdots - C_{n-1}(t-a)^{n-1}\}\, dt.$$

This function has the needed derivatives and satisfies conditions (2a) and (2b) because

$$\tilde{\eta}^{(k)}(x) = \int_a^x \frac{(x-t)^{n-k-1}}{(n-k-1)!} \{M_n(t) - C_0 - C_1(t-a) - \cdots - C_{n-1}(t-a)^{n-1}\}\, dt, \qquad (k = 0, 1, 2, \cdots, n-1).$$

The equations of the form $\tilde{\eta}^{(k)}(a) = 0$ are obvious and the equations of the form, $\tilde{\eta}^{(k)}(b) = 0$ follow from equation (4). Hence

$$\int_a^b M_n(x)\tilde{\eta}^{(n)}(x)\, dx = 0$$

or

$$\int_a^b \{M_n(x) - C_0 - C_1(x-a) - \cdots - C_{n-1}(x-a)^{n-1}\} M_n(x)\, dx = 0$$

but, from condition (4), this equation can be put in the form

$$\int_a^b \{M_n(x) - C_0 - C_1(x-a) - \cdots - C_{n-1}(x-a)^{n-1}\}^2\, dx = 0$$

and our assertion follows from this.

We have shown that the function $\bar{y}(x)$ which furnishes the functional (1) with an extremum and every function for which the first variation vanishes satisfy the integral equation

$$f_{y^{(n)}} - \int_a^x f_{y^{(n-1)}}\,dx + \int_a^x\int_a^x f_{y^{(n-2)}}\,dx\,dx + \cdots + (-1)^n \underbrace{\int_a^x\int_a^x\cdots\int_a^x}_{n \text{ times}} f_y\,dx\,dx\cdots dx = C_0 + C_1(x-a) + \cdots + C_{n-1}(x-a)^{n-1}$$

in which the C_i are constants. In every interval in which the function $\bar{y}(x)$ has a continuous derivative of order n, it satisfies the equation

$$\frac{d}{dx}\left(\cdots\frac{d}{dx}\left\{\frac{d}{dx}\left[\frac{d}{dx}f_{y^{(n)}} - f_{y^{(n-1)}}\right] + f_{y^{(n-2)}}\right\} + \cdots + (-1)^{n-1}f_{y'}\right) + (-1)^n f_y = 0.$$

If the function $\bar{y}(x)$ has a derivative of order $2n$, then we obtain the differential equation

$$f_y - \frac{d}{dx}f_{y'} + \frac{d^2}{dx^2}f_{y''} - \frac{d^3}{dx^3}f_{y'''} + \cdots + (-1)^n\frac{d^n}{dx^n}f_{y^{(n)}} = 0 \quad (5)$$

which is a generalization of the Euler equation

$$f_y - \frac{d}{dx}f_{y'} = 0.$$

We can obtain the Euler equation from equation (5) by setting $n = 1$.

An equation analogous to equation (5) can be obtained for a multiple integral. We restrict ourselves to stating the result for the case of a double integral and for simplicity take $n = 2$. For this case we have the functional

$$J[u] = \iint_D f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\,dx\,dy. \quad (6)$$

We suppose that the function f has continuous derivatives of the third order in all its arguments.

If the function $\bar{u}(x, y)$ has continuous derivatives of the fourth order and gives the functional (6) an extremum then it must satisfy the partial differential equation

$$f_u - \frac{\partial}{\partial x} f_{u_x} - \frac{\partial}{\partial y} f_{u_y} + \frac{\partial^2}{\partial x^2} f_{u_{xx}} + \frac{\partial^2}{\partial x \partial y} f_{u_{xy}} + \frac{\partial^2}{\partial y^2} f_{u_{yy}} = 0.$$

24. THE ELEMENTARY MULTIPLIER RULE; THE GRAM DETERMINANT

In the two sections that follow we shall be concerned with calculus of variations problems for *conditioned extrema.* These problems are generalizations of the problems of differential calculus for extrema with side conditions; the well-known *Lagrange multiplier rule* was introduced for the solution of such problems. Because we shall need to use this rule, it is expedient to recall it. We shall use the method of Bliss.*

Consider the problem of an extremum for the function

$$u = \phi_0(x_1, x_2, \cdots, x_k) \tag{1}$$

with the conditions

$$\phi_i(x_1, x_2, \cdots, x_k) = 0, \qquad (i = 1, 2, \cdots, j < k). \tag{2}$$

All the functions are assumed to be continuously differentiable in some region of k-dimensional space. Suppose that in some interior point

$$(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k) \tag{3}$$

of this region that satisfies condition (2), the function (1) has an extremum $\bar{u}$.

We assert that under these conditions the rank of the matrix

$$\begin{pmatrix} \bar{\phi}_{0x_1} & \bar{\phi}_{0x_2} & & \bar{\phi}_{0x_k} \\ \bar{\phi}_{1x_1} & \bar{\phi}_{1x_2} & & \bar{\phi}_{1x_k} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{\phi}_{jx_1} & \bar{\phi}_{jx_2} & & \bar{\phi}_{jx_k} \end{pmatrix} \tag{4}$$

*G. A. Bliss, *Lectures on the Calculus of Variations* (University of Chicago Press, Chicago, 1946), p. 201.

is less than $j + 1$. Here we have used the notation

$$\bar{\psi} = \psi(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k), \qquad \phi_{ix_r} = \frac{\partial \phi_i}{\partial x_r}.$$

To prove that the rank of this matrix is less than $j + 1$, assume the contrary. Then both the matrix

$$\begin{pmatrix} \bar{\phi}_{0x_1} & \bar{\phi}_{0x_2} & & \bar{\phi}_{0x_k} & -1 \\ \bar{\phi}_{1x_1} & \bar{\phi}_{1x_2} & & \bar{\phi}_{1x_k} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{\phi}_{jx_1} & \bar{\phi}_{jx_2} & & \bar{\phi}_{jx_k} & 0 \end{pmatrix}$$

and the matrix (4) have rank $j + 1$. Hence the system of equations

$$\begin{aligned} \phi_0(x_1, x_2, \cdots, x_k) - h &= \bar{u} \\ \phi_1(x_1, x_2, \cdots, x_k) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \phi_j(x_1, x_2, \cdots, x_k) &= 0 \end{aligned}$$

can be solved for $j + 1$ of the variables $x_1, x_2, \cdots, x_k$ in a sufficiently small neighborhood of the point

$$x_1 = \bar{x}_1, \qquad x_2 = \bar{x}_2, \cdots, x_k = \bar{x}_k, \qquad h = 0$$

if $|h|$ is small enough, whatever values the remaining k-j-1 variables have sufficiently close to the corresponding $\bar{x}_r$. It follows from this that in a sufficiently small neighborhood of the point (3) we can find points for which the function (1) takes values both greater than $\bar{u}$ and less than $\bar{u}$. This implies that the function (1) does not have an extremum that satisfies the side conditions, and proves our assertion.

If the rank of the matrix (4) must be less than $j + 1$ at the point (3) where the extremum with side conditions is attained, then at this point the $j + 1$ k-dimensional vectors

$$\text{grad } \phi_0, \text{ grad } \phi_1, \cdots, \text{ grad } \phi_j$$

must be linearly dependent. In other words, there must exist constants

$$\lambda_0, \lambda_1, \cdots, \lambda_j \qquad (\lambda_0^2 + \lambda_1^2 + \cdots + \lambda_j^2) \neq 0$$

such that the gradient of the function

$$\lambda_0\phi_0 + \lambda_1\phi_1 + \cdots + \lambda_j\phi_j$$

is zero at the point (3). This is the elementary multiplier rule.

If λ_0 is not zero, we can set $\lambda_0 = 1$. This is possible if and only if the rank of the matrix

$$\begin{pmatrix} \bar{\phi}_{1x_1} & \bar{\phi}_{1x_2} & \bar{\phi}_{1x_k} \\ \bar{\phi}_{2x_1} & \bar{\phi}_{2x_2} & \bar{\phi}_{2x_k} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \bar{\phi}_{jx_1} & \bar{\phi}_{jx_2} & \bar{\phi}_{jx_k} \end{pmatrix}$$

is equal to j. We shall need criteria for the linear independence of a system of vector functions, and we introduce the criteria which are suitable for our needs. Consider a system of real piecewise-continuous vector functions

$$G_i(x) = \{g_{i1}(x), g_{i2}(x), \cdots, g_{in}(x)\}, \qquad (a \leq x \leq b), \quad (i = 1, 2, \cdots, m). \tag{5}$$

Set

$$(G_i, G_j) = \int_a^b \sum_{k=1}^{n} g_{ik}(x)g_{jk}(x)\,dx, \qquad (i, j = 1, 2, \cdots, m).$$

The vector functions $G_i(x)$ are linearly independent if and only if the Gram determinant

$$\begin{vmatrix} (G_1, G_1) & (G_1, G_2) & (G_1, G_m) \\ (G_2, G_1) & (G_2, G_2) & (G_2, G_m) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ (G_m, G_1) & (G_m, G_2) & (G_m, G_m) \end{vmatrix} \tag{6}$$

is different from zero.

PROOF: If the vector functions (5) are linearly dependent, then there exist constants λ_i not all zero for which

$$\lambda_1 g_{1k}(x) + \lambda_2 g_{2k}(x) + \cdots + \lambda_m g_{mk}(x) = 0, \qquad (a \leq x \leq b), \quad (k = 1, 2, \cdots, n).$$

Hence

$$\lambda_1 \int_a^b g_{ik}(x)g_{1k}(x)\,dx + \lambda_2 \int_a^b g_{ik}(x)g_{2k}(x)\,dx + \cdots + \lambda_m \int_a^b g_{ik}(x)g_{mk}(x)\,dx = 0,$$
$$(k = 1, 2, \cdots, n), \qquad (i = 1, 2, \cdots, m)$$

which implies

$$\lambda_1(G_i, G_1) + \lambda_2(G_i, G_2) + \cdots + \lambda_m(G_i, G_m) = 0, \qquad (i = 1, 2, \cdots, m). \quad (7)$$

By hypothesis, not all λ_j are zero; thus the determinant of this system must be zero. This shows that the vanishing of the determinant (6) follows from the linear dependence of the vector functions (5).

Suppose that determinant (6) is zero. Then the system (7) has a nontrivial solution $\lambda_1, \lambda_2, \cdots, \lambda_m$; if we multiply equations (7) by λ_i and sum with respect to i we find that

$$\sum_{i,j=1}^{m} (G_i, G_j)\lambda_i\lambda_j = 0.$$

Now set

$$\Phi(x) = \sum_{k=1}^{n} \{\lambda_1 g_{1k}(x) + \lambda_2 g_{2k}(x) + \cdots + \lambda_m g_{mk}(x)\}^2$$
$$= \sum_{k=1}^{n} \sum_{i,j=1}^{m} \lambda_i\lambda_j g_{ik}(x)g_{jk}(x)$$

from which

$$\int_a^b \Phi(x)\,dx = \sum_{i,j=1}^{m} (G_i, G_j)\lambda_i\lambda_j.$$

Hence

$$\int_a^b \Phi(x)\,dx = 0$$

which implies that everywhere on the interval $[a, b]$

$$\lambda_1 g_{1k}(x) + \lambda_2 g_{2k}(x) + \cdots + \lambda_m g_{mk}(x) = 0, \qquad (k = 1, 2, \cdots, n).$$

This demonstrates that the linear dependence of the vector functions (5) follows from the vanishing of the determinant (6).

25. THE ISOPERIMETRIC PROBLEM

In its original or narrow sense, the isoperimetric problem is stated as follows. Among all closed plane curves of given length, find the one that bounds the largest area. The well-known solution of this problem is a circle. Both this solution and its three-space analogue were known in ancient times.

Analytically, the isoperimetric problem is this: Find the maximum of the functional in parametric form

$$\tfrac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})\, dt$$

such that

$$\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = L$$

where L is given and such that

$$x(t_1) = x(t_2), \qquad y(t_1) = y(t_2).$$

The problem is that of finding an extremum for some functional subject to the condition that another functional has a given value. It is customary to call all problems of this kind *isoperimetric* problems.

We consider the fundamental problem for functionals that are simple integrals (not multiple) and curves with fixed end points.*

We restate the problem: Among all piecewise-smooth vector functions $Y(x) = \{y_1(x), \cdots, y_n(x)\}$ that assume given values at the end points of the interval $[a, b]$, find the one that gives an extremum to the functional

$$J_0[Y] = \int_a^b f_0(x, Y, Y')\, dx$$

*In Section 32 of the Appendix an important isoperimetric problem in which the natural condition is imposed at one end point is investigated.

and has the property that

$$J_i[Y] = \int_a^b f_i(x, Y, Y')\,dx = L_i, \qquad (i = 1, 2, \cdots, k). \quad (1)$$

At the point (x, Y) in some region G of $x, y_1\, y_2 \cdots y_n$-space the functions $f_j(x, Y, Z)$, $(j = 0, 1, 2, \cdots, k)$ are defined and have continuous derivatives of the second order in all their arguments and the vector Z assumes any finite value.

We note that in this problem, admissible vector functions satisfy both conditions (1) and the end point conditions. The set of admissible vector functions may thus be empty for a given set of values L_i, $(i = 1, 2, \cdots, k)$.

We restrict ourselves to one of Euler's theorems which is called the *multiplier rule for the isoperimetric problem.*

THEOREM: *If a piecewise-smooth curve $Y = \bar{Y}(x)$ that lies (with the possible exception of its end points) in G gives the functional $J_0[Y]$ an extremum with conditions* (1) *then there exist constants λ_j, $(\lambda_0^2 + \lambda_1^2 + \cdots + \lambda_k^2 = 1)$ such that $Y = \bar{Y}(x)$ is an ordinary extremal (without side conditions) for the functional*

$$\int_a^b (\lambda_0 f_0 + \lambda_1 f_1 + \cdots + \lambda_k f_k)\,dx \qquad (2)$$

*that is, an extremal that corresponds to an unconstrained variation.**

The value of this theorem lies in the fact that it enables us to find all the curves that can be solutions of the isoperimetric problem. To do this we must consider the λ_i as unknown parameters and find the Euler equations. In the integration of these equations $2n$ unknown constants will appear so that there will be $2n + k$ constants in all. We have k equations (1) and $2n$ boundary conditions to determine these constants.

To prove the multiplier rule we introduce the functions

$$[\bar{f}_{jr}] = f_{jy_r'}(x, \bar{Y}, \bar{Y}') - \int_a^x f_{jy_r}(x, \bar{Y}, \bar{Y}')\,dx - c_{jr},$$
$$(j = 0, 1, 2, \cdots, k; \quad r = 1, 2, \cdots, n)$$

*Such an extremal may be smooth or broken and is also referred to as a free extremal. Unconstrained variations are also called free variations.

in which the constants c_{jr} are such that

$$\int_a^b [\bar{f}_{jr}]\,dx = 0, \qquad (j = 0, 1, \cdots, k; \quad r = 1, 2, \cdots, n). \tag{3}$$

We also introduce the vector functions

$$F_j(x) = \{[\bar{f}_{j1}], [\bar{f}_{j2}], \cdots, [\bar{f}_{jn}]\}, \qquad (j = 0, 1, 2, \cdots, k).$$

The multiplier rule will be proved if we show that the vector functions

$$F_0(x), F_1(x), \cdots, F_k(x) \tag{4}$$

are linearly dependent; linear dependence of these vector functions shows that there exist constants λ_j, $(\lambda_0^2 + \lambda_1^2 + \cdots + \lambda_k^2 = 1)$ for which

$$\lambda_0[\bar{f}_{0r}] + \lambda_1[\bar{f}_{1r}] + \cdots + \lambda_k[\bar{f}_{kr}] = 0, \qquad (r = 1, 2, \cdots, n).$$

But these relations can be written in the form

$$\Phi_{y_r'}(x, \bar{Y}, \bar{Y}') - \int_a^x \Phi_{y_r}(x, \bar{Y}, \bar{Y}')\,dx = \Gamma_r, \qquad (r = 1, 2, \cdots, n)$$

where

$$\Phi = \lambda_0 f_0 + \lambda_1 f_1 + \cdots + \lambda_k f_k,$$
$$\Gamma_r = \lambda_0 c_{0r} + \lambda_1 c_{1r} + \cdots + \lambda_k c_{kr}, \qquad (r = 1, 2, \cdots, n)$$

and consequently the linear dependence of the vector functions (4) shows that $Y = \bar{Y}(x)$ is an extremal without side conditions for the functional (2).

To prove that the vector functions (4) are linearly dependent in accordance with the statements in Section 24 we must show that the Gram determinant

$$\mathfrak{G}(F_0, F_1, \cdots, F_k) = \begin{vmatrix} (F_0, F_0) & (F_0, F_1) & \cdots & (F_0, F_k) \\ (F_1, F_0) & (F_1, F_1) & \cdots & (F_1, F_k) \\ \cdots & \cdots & \cdots & \cdots \\ (F_k, F_0) & (F_k, F_1) & \cdots & (F_k, F_k) \end{vmatrix} \tag{5}$$

where

$$(F_i, F_j) = \int_a^b \{[\bar{f}_{i1}][\bar{f}_{j1}] + [\bar{f}_{i2}][\bar{f}_{j2}] + \cdots + [\bar{f}_{in}][\bar{f}_{jn}]\}\,dx$$

must be zero.

We shall assume that

$$\mathfrak{G}(F_1, F_2, \cdots, F_k) \neq 0 \tag{6}$$

because otherwise the vector functions

$$F_1, F_2, \cdots, F_k$$

would be linearly dependent.

We now vary the vector function $\bar{Y}(x)$ by setting

$$\tilde{Y}(x) = \bar{Y}(x) + \epsilon_0 \int_a^x F_0(x)\, dx + \epsilon_1 \int_a^x F_1(x)\, dx + \cdots + \epsilon_k \int_a^x F_k(x)\, dx \tag{7}$$

where the ϵ_j are sufficiently small in absolute value. The components of equation (7) are

$$\tilde{y}_r(x) = \bar{y}_r(x) + \epsilon_0 \int_a^x [\bar{f}_{0r}]\, dx + \epsilon_1 \int_a^x [\bar{f}_{1r}]\, dx + \cdots + \epsilon_k \int_a^x [\bar{f}_{kr}]\, dx.$$

The difference

$$\tilde{Y}(x) - \bar{Y}(x)$$

vanishes for arbitrary ϵ_j at $x = a$, and it follows from (3) that this difference is zero for $x = b$ also. The vector function $\tilde{Y}(x)$ thus satisfies the boundary conditions for arbitrary ϵ_j. On the other hand if we set

$$J_j[\tilde{Y}] = \phi_j(\epsilon_0, \epsilon_1, \cdots, \epsilon_k), \qquad (j = 0, 1, 2, \cdots, k)$$

then

$$\phi_0(0, 0, \cdots, 0) = J_0[\bar{Y}],$$

$$\phi_i(0, 0, \cdots, 0) = L_i, \qquad (i = 1, 2, \cdots, k)$$

and

$$\bar{\phi}_{j\epsilon_i} \equiv \left(\frac{\partial \phi_j}{\partial \epsilon_i}\right)_{\epsilon_0 = \epsilon_1 = \cdots = \epsilon_k = 0} = \int_a^b \sum_{r=1}^{n} \left\{ \bar{f}_{jy_r} \int_a^x [\bar{f}_{ir}]\, dx + \bar{f}_{jy'_r}[\bar{f}_{ir}] \right\} dx$$

$$= \int_a^b \sum_{r=1}^{n} \left\{ \bar{f}_{jy_r'} - \int_a^x \bar{f}_{jy_r}\, dx - c_{jr} \right\} [\bar{f}_{ir}]\, dx$$

$$= \int_a^b \sum_{r=1}^{n} [\bar{f}_{jr}][\bar{f}_{ir}]\, dx = (F_j, F_i)$$

so that

$$|\bar{\phi}_{j\epsilon_i}|_{i,j=0}^{k} = \mathfrak{G}(F_0, F_1, \cdots, F_k) \tag{8}$$

$$|\bar{\phi}_{j\epsilon_i}|_{i,j=1}^{k} = \mathfrak{G}(F_1, F_2, \cdots, F_k). \tag{9}$$

Relation (9) and assumption (6) provide that in a sufficiently small neighborhood of $\epsilon_0 = 0$ the equations

$$\phi_j(\epsilon_0, \epsilon_1, \cdots, \epsilon_k) = L_j, \qquad (j = 1, 2, \cdots, k) \tag{10}$$

define the quantities $\epsilon_1, \epsilon_2, \cdots, \epsilon_k$ as continuously differentiable functions of ϵ_0 that vanish for $\epsilon_0 = 0$. Because the function $\phi_0(\epsilon_0, \epsilon_1, \cdots, \epsilon_k)$ has an extremum for condition (10) at $\epsilon_0 = \epsilon_1 = \cdots = \epsilon_k = 0$ the elementary multiplier rule (see page 109) insures that the rank of the matrix

$$\begin{pmatrix} \bar{\phi}_{0\epsilon_0} & \bar{\phi}_{0\epsilon_1} & & \bar{\phi}_{0\epsilon_k} \\ \bar{\phi}_{1\epsilon_0} & \bar{\phi}_{1\epsilon_1} & & \bar{\phi}_{1\epsilon_k} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{\phi}_{k\epsilon_0} & \bar{\phi}_{k\epsilon_1} & & \bar{\phi}_{k\epsilon_k} \end{pmatrix}$$

must be less than $k + 1$; that is, the determinant

$$|\bar{\phi}_{j\epsilon_i}|_{i,j=0}^{k}$$

must be equal to zero so that, from relation (8), the Gram determinant (5) must be zero. This concludes the proof.

26. THE LAGRANGE PROBLEM

A more complicated problem with side conditions is called the *Lagrange problem*. It consists of finding extrema for the functional

$$\int_a^b f(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')\, dx$$

with side conditions

$$g_i(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n') = 0, \qquad (i = 1, 2, \cdots, k < n)$$

and such that certain boundary conditions are satisfied.

We shall consider only the simplest case: $n = 2, k = 1$, and limit ourselves to deriving the multiplier rule.

We consider the functional

$$J[y, z] = \int_a^b f(x, y, z, y', z')\, dx \tag{1}$$

subject to the relation

$$g(x, y, z, y', z') = 0. \tag{2}$$

As usual, the functions f and g are assumed to have continuous derivatives of the second order for (x, y, z) in some region G; the derivatives y', z' have arbitrary finite values.

We say that condition (2) is *holonomic* if equation (2) does not contain y', z' or if it can be reduced to a form not containing y', z'. Otherwise we say that it is *nonholonomic*. If the side condition is holonomic, it can be put in the form

$$h(x, y, z) = 0 \tag{3}$$

and we shall assume that

$$h_z(x, y, z) \neq 0. \tag{4}$$

In this case we shall consider the boundary conditions

$$y(a) = a_1, \qquad y(b) = b_1.$$

For the nonholonomic case of condition (2) we shall assume that

$$g_{z'}(x, y, z, y', z') \neq 0 \tag{5}$$

and consider the boundary conditions

$$y(a) = a_1, \qquad y(b) = b_1, \qquad z(b) = b_2 \tag{6}$$

that is, we consider y with both end points fixed, suppose that z has a fixed right-hand end point and impose the natural condition on the left-hand end point.

This choice of boundary conditions is motivated by the fact that

the side condition must define one of the functions y, z in terms of x and the other. In the case of the holonomic condition (3), condition (4) permits us to consider z as a function of x and y. Hence the values of the function y at points a and b define the function z at these points. The nonholonomic case (2) with condition (5) can be considered as a differential equation for z if the function y is known. Integration of this equation introduces one arbitrary constant which we can choose to satisfy one condition, for example, $z(b) = b_2$.

For the nonholonomic condition (2) the Lagrange problem can be stated for fixed end points

$$y(a) = a_1, \quad y(b) = b_1, \quad z(a) = a_2, \quad z(b) = b_2. \quad (6')$$

Then, as before, if we consider relation (2) as a definition of $z(x)$ in terms of $y(x)$, it is necessary to find a function $y(x)$ that satisfies the conditions

$$y(a) = a_1, \quad y(b) = b_1$$

for which the function $z(x)$ defined by equation (2) for a proper choice of the constant of integration will satisfy the conditions

$$z(a) = a_2, \quad z(b) = b_2.$$

However, such a statement of the problem, even in the simplest cases, may have no meaning. For example, suppose the side condition is

$$z' = z^2 + y'^2$$

then if we have the conditions

$$y(a) = 1, \quad y(b) = 1, \quad z(b) = 0$$

the condition $z(a) = \alpha > 0$ at the left-hand end point will not be realized; the condition $\alpha = 0$ yields only one curve:

$$y = 1, \quad z = 0.$$

Nevertheless, the case may arise in which the Lagrange problem must be solved for conditions (6'). Consider the problem of an extremum for the functional

$$J[y] = \int_a^b f(x, y, y', y'')\, dx$$

for conditions

$$y(a) = a_1, \qquad y'(a) = a_2$$
$$y(b) = b_1, \qquad y'(b) = b_2.$$

If we set $z = y'$ then this problem can be treated as the Lagrange problem of finding an extremum for the functional

$$\int_a^b f(x, y, z, z')\, dx$$

with conditions (6′) and the nonholonomic side condition.

We restrict ourselves to considering conditions (6) and remark that the multiplier rule we derive remains in force even for fixed end points if the corresponding problem has meaning; the rule can be generalized to the case of arbitrary $n > 2$ and arbitrary $k < n$.

THEOREM: *If twice continuously differentiable functions* $\bar{y}(x)$, $\bar{z}(x)$ *furnish an extremum for the functional* (1) *for condition* (2) *and end conditions* (6) *and if*

$$\bar{g}_{z'} \neq 0, \qquad a \leq x \leq b \tag{7}$$

then there exists a multiplier $\lambda = \lambda(x)$ *such that the vector function* $\{\bar{y}(x), \bar{z}(x)\}$ *is an extremal without side condition for the functional*

$$\int_a^b \{f(x, y, z, y', z') + \lambda(x)g(x, y, z, y', z')\}\, dx$$

and satisfies the condition

$$(f_{z'} + \lambda g_{z'})|_{x=a} = 0. \tag{8}$$

For the proof we introduce an arbitrary continuously differentiable function $\eta(x)$ that satisfies the conditions

$$\eta(a) = \eta(b) = 0$$

and set

$$y(x) = \bar{y}(x) + \epsilon\eta(x)$$

where ϵ is a parameter. Condition (2) becomes

$$g(x, \bar{y}(x) + \epsilon\eta(x), z, \bar{y}'(x) + \epsilon\eta'(x), z') = 0$$

which is a differential equation of the first order for z. For $\epsilon = 0$, this differential equation has a solution $\bar{z}(x)$ in the interval $[a, b]$ that satisfies the condition

$$\bar{z}(b) = b_2.$$

Condition (7) and general theorems of Section 8 provide that this equation has a solution $z(x, \epsilon)$ for $|\epsilon|$ sufficiently small. This solution has a smooth derivative with respect to ϵ and satisfies the condition

$$z(b, \epsilon) = b_2$$

where

$$z(x, 0) = \bar{z}(x).$$

The family of vector functions with components

$$y = \bar{y}(x) + \epsilon\eta(x), \qquad z = z(x, \epsilon)$$

satisfies requirements (2) and (6). By hypothesis the function

$$\phi(\epsilon) = J[\bar{y}(x) + \epsilon\eta(x), z(x, \epsilon)]$$

has an extremum for $\epsilon = 0$. Hence

$$\phi'(0) = 0.$$

It follows that

$$\int_a^b \{\bar{f}_y\eta + \bar{f}_{y'}\eta' + \bar{f}_z z_\epsilon(x, 0) + \bar{f}_{z'} z_{x\epsilon}(x, 0)\}\, dx = 0.$$

Because the equation

$$\bar{g}_y\eta + \bar{g}_{y'}\eta' + \bar{g}_z z_\epsilon(x, 0) + \bar{g}_{z'} z_{x\epsilon}(x, 0) = 0$$

holds, we may write

$$\int_a^b \{(\bar{f}_y + \lambda\bar{g}_y)\eta + (\bar{f}_{y'} + \lambda\bar{g}_{y'})\eta' + (\bar{f}_z + \lambda\bar{g}_z)z_\epsilon(x, 0) + (\bar{f}_{z'} + \bar{g}_{z'})z_{x\epsilon}(x, 0)\}\, dx = 0$$

for arbitrary $\lambda = \lambda(x)$.

In what follows we shall assume that $\lambda(x)$ is a continuously differentiable function. Hence it is possible to integrate by parts;

this yields

$$\int_a^b \left\{ \left[(\bar{f}_y + \lambda \bar{g}_y) - \frac{d}{dx}(\bar{f}_{y'} + \lambda \bar{g}_{y'}) \right] \eta(x) + \left[(\bar{f}_z + \lambda \bar{g}_z) - \frac{d}{dx}(\bar{f}_{z'} + \lambda \bar{g}_{z'}) \right] z_\epsilon(x, 0) \right\} dx - (\bar{f}_{z'} + \lambda \bar{g}_{z'}) z_\epsilon(x, 0)|_{x=a} = 0. \tag{9}$$

The integrated term does not contain $\eta(x)$ because

$$\eta(a) = \eta(b) = 0.$$

The integrated term with $z_\epsilon(x, 0)$ also vanishes for $x = b$ because $z(b, \epsilon) = b_2$ does not depend on ϵ. We now impose the condition that $\bar{f}_{z'} + \lambda \bar{g}_{z'}$ must vanish at $x = a$. This is the first condition on $\lambda(x)$; condition (7) allows us to impose it. We may also impose the condition that

$$(\bar{f}_z + \lambda \bar{g}_z) - \frac{d}{dx}(\bar{f}_{z'} + \lambda \bar{g}_{z'}) = 0 \tag{10}$$

on the interval (a, b). This is a linear differential equation of the first order for λ and condition (7) provides that the coefficient of λ' does not vanish. The function $\lambda(x)$ is defined uniquely by this equation because its value at $x = a$ has already been established. For the given multiplier $\lambda(x)$, condition (9) becomes

$$\int_a^b \left\{ (\bar{f}_y + \lambda \bar{g}_y) - \frac{d}{dx}(\bar{f}_{y'} + \lambda \bar{g}_{y'}) \right\} \eta(x)\, dx = 0.$$

Hence it follows from the arbitrary character of $\eta(x)$ that

$$(\bar{f}_y + \lambda \bar{g}_y) - \frac{d}{dx}(\bar{f}_{y'} + \lambda \bar{g}_{y'}) = 0. \tag{11}$$

Equations (11) and (10) show that if the pair of functions $\bar{y}(x), \bar{z}(x)$ furnish an extremum they are solutions of the system of equations

$$(f_y + \lambda g_y) - \frac{d}{dx}(f_{y'} + \lambda g_{y'}) = 0$$

$$(f_z + \lambda g_z) - \frac{d}{dx}(f_{z'} + \lambda g_{z'}) = 0 \tag{12}$$

that is, they are components of an extremal for the functional

$$\int_a^b \{f(x, y, z, y', z') + \lambda(x)g(x, y, z, y', z')\}\, dx$$

for unconstrained variations.* Because the boundary condition (8) is also obtained the theorem is proved. In applying this theorem we must solve equations (2), (12) for the functions $y(x)$, $z(x)$, $\lambda(x)$. It is not difficult to show that this is a system of the fourth order; its solution introduces four constants which are determined by conditions (6) and (8).

Let us turn now to the holonomic condition (3). Suppose that a pair of twice continuously differentiable functions $\bar{y}(x), \bar{z}(x)$ furnish an extremum for the functional (1) with side conditions (3) and boundary conditions

$$y(a) = a_1, \qquad y(b) = b_1 \tag{13}$$

and suppose that

$$\bar{h}_z \neq 0, \qquad (a \leq x \leq b).$$

Since $\bar{h}_z \neq 0$, equation (3) defines a twice continuously differentiable function $z = \omega(x, y)$ in some neighborhood of the curve $y = \bar{y}(x)$; this function coincides with $\bar{z}(x)$ for $y = \bar{y}(x)$. If we replace z by $\omega(x, y)$ in the functional (1) we obtain a new functional

$$J[y] = \int_a^b f(x, y, \omega(x, y), y', \omega_x(x, y) + y'\omega_y(x, y))\, dx.$$

If we use boundary conditions (13) and no side condition, the function $\bar{y}(x)$ furnishes an extremum for this new functional. Hence we have

$$\bar{F}_y - \frac{d}{dx}\bar{F}_{y'} = 0 \tag{14}$$

where

$$F(x, y, y') = f(x, y, \omega(x, y), y', \omega_x(x, y) + y'\omega_y(x, y)).$$

Making use of this explicit expression for the function F we may

*These are variations without side conditions. The term "unconstrained variations" was first introduced in the statement of the theorem in Section 25; the term "free variations" is sometimes used synonymously.

write equation (14) in the form

$$\bar{f}_y + \bar{f}_z\bar{\omega}_y + \bar{f}_{z'}\{\bar{\omega}_{xy} + \bar{y}'\bar{\omega}_{yy}\} - \frac{d}{dx}\{\bar{f}_{y'} + \bar{f}_{z'}\bar{\omega}_y\} = 0$$

or

$$\bar{f}_y + \bar{f}_z\bar{\omega}_y - \frac{d}{dx}\bar{f}_{y'} - \bar{\omega}_y\frac{d}{dx}\bar{f}_{z'} = 0 \tag{15}$$

because

$$\frac{d}{dx}\bar{\omega}_y = \bar{\omega}_{xy} + \bar{y}'\bar{\omega}_{yy}.$$

If we notice that

$$\omega_y(x, y) = -\frac{h_y(x, y, z)}{h_z(x, y, z)}$$

we may write equation (15) in the form

$$\left(\bar{f}_y - \frac{d}{dx}\bar{f}_{y'}\right) - h_y(x, \bar{y}, \bar{z})\frac{\bar{f}_z - \frac{d}{dx}\bar{f}_{z'}}{h_z(x, \bar{y}, \bar{z})} = 0. \tag{16}$$

The ratio

$$\frac{\bar{f}_z - \frac{d}{dx}\bar{f}_{z'}}{h_z(x, \bar{y}, \bar{z})}$$

is a function of x. We denote it by $\lambda(x)$ and obtain from equation (16) two identities:

$$\bar{f}_z - \frac{d}{dx}\bar{f}_{z'} = -\lambda(x)h_z(x, \bar{y}, \bar{z})$$

$$\bar{f}_y - \frac{d}{dx}\bar{f}_{y'} = -\lambda(x)h_y(x, \bar{y}, \bar{z})$$

from which it follows that the functions $\bar{y}(x), \bar{z}(x)$ satisfy the Euler equation for the functional

$$\int_a^b \{f(x, y, z, y', z') + \lambda(x)h(x, y, z)\}\,dx.$$

Thus we have proved the multiplier rule for the holonomic case.

EXERCISES

1. Find a function that may furnish an extremum for

$$\int_0^{\pi/4} (y^2 - y'^2)\, dx; \qquad y(0) = 0$$

if the right-hand end point lies on the line $x = \frac{1}{4}\pi$.

2. Making use of only the fundamental necessary condition, $\delta J = 0$, find a curve that may furnish $J[y]$ with an extremum where

$$J[y] = \int_0^{x_1} \frac{\sqrt{1 + y'^2}}{y}\, dx; \qquad y(0) = 0$$

if the end point (x_1, y_1) lies on the circle $(x - 9)^2 + y^2 = 9$.

3. Find the transversality condition for

$$\int_{x_0}^{x_1} A(x, y) \exp(\arctan y') \sqrt{1 + y'^2}\, dx; \qquad A(x, y) \neq 0.$$

4. Find the Euler (Lagrange-Ostrogradsky) equation for

$$\iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2\right] dx\, dy.$$

5. Find extremals for

$$\int_{x_0}^{x_1} (16y^2 - y''^2 + x^2)\, dx.$$

6. Find extremals for

$$\int_{x_0}^{x_1} (2xy + y'''^2)\, dx.$$

7. Find extremals for the functional

$$\int_0^1 (y'^2 + x^2)\, dx$$

under the conditions

$$\int_0^1 y^2\,dx = 2, \qquad y(0) = 0, \qquad y(1) = 0.$$

8. Find an extremal for

$$\int_0^1 (y'^2 + z'^2 - 4xz' - 4z)\,dx$$

where

$$\int_0^1 (y'^2 - xy' - z'^2)\,dx = 2, \qquad y(0) = 0,$$
$$z(0) = 0, \qquad y(1) = 1, \qquad z(1) = 1$$

4

Direct Methods of the Calculus of Variations

27. THE CONCEPT OF DIRECT METHODS

Classical methods reduce the fundamental question of the existence of a solution for an arbitrary calculus of variations problem to the question of the existence of solutions of differential equations. A solution is sought in a region for given boundary conditions (a solution in the whole region). Only in rare instances does the theory of differential equations provide proof of the existence of such solutions. Consequently, mathematicians have sought other approaches to variation problems and created the so-called *direct methods*.

The direct methods of the calculus of variations have proved to be useful in the theory of differential equations. If a differential equation can be treated as the Euler equation for some functional and if we can use a direct method to establish that this functional has an extremum in a class of functions differentiable a sufficient number of times, then we shall have proved that this differential equation has a solution for the considered boundary conditions. A direct method consists in the construction of a sequence of functions that converges to the desired function; thus the direct method both establishes the existence of a solution and provides a way of constructing an approximate solution.

Riemann developed the first important and systematic idea of going from boundary value problems of differential equations to variational boundary problems. His investigation was needed to prove that there may exist a solution of Dirichlet's problem for an arbitrary plane region bounded by a contour; that is, it was needed to prove the existence of a function $u(x, y)$ that has continuous

derivatives of the second order with respect to x and y in a region of the plane and that satisfies the *Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

and that coincides with a given continuous function on the boundary of the region. Laplace's equation (1) is the Euler equation for the functional

$$D[u] = \iint_S \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dx\, dy$$

which assumes only nonnegative values. Because of this fact Riemann considered as obvious the existence of a minimum for this functional, and hence assumed the existence of a solution of equation (1) satisfying the given boundary conditions. Riemann encountered this argument (which was attributed to Gauss and Thompson) while he was a student attending the lectures of Dirichlet; and it was Riemann who called it the *Dirichlet principle.*

Weierstrass subjected Riemann's reasoning to criticism: The fact that the functional is bounded from below implies only that it has a greatest lower bound; it does not insure that this bound is achieved for a function in the class under consideration; that is, it does not insure that this lower bound is a minimum. Weierstrass cited a calculus of variations problem of this kind for which there is no solution, and showed that it was impossible to remove this objection.* Hence interested mathematicians sought to secure a foundation for the remarkable results Riemann obtained with the aid of Dirichlet's principle. The objections Weierstrass had made were of a general nature and were not completely removed so that the principle of Dirichlet, which concerned a special functional, could not be justified immediately. From time to time a little was done. Hilbert took the essential steps toward the justification of the Dirichlet principle; a whole constellation of mathematicians continued his researches. Their investigations led both to the

*See Appendix, Section A–7.

justification of Dirichlet's principle in a modified form and to the invention of direct methods in the calculus of variations. N. N. Bogolyubov, N.M. Krylov, M. A. Lavrent'ev, and L.A. Lyusternik have made progress in the development of these methods.

To describe the fundamental steps involved in a direct method we shall assume that the problem involves minimizing a functional J_C; here C is a member of some collection $\mathfrak{M}$ of curves. For the problem to have meaning it is necessary to assume that the functional J_C is finite and also that

$$\inf_{C \in \mathfrak{M}} J_C = \mu > -\infty.$$

According to the definition of lower bound, there exists a sequence (Σ) of curves

$$C_1, C_2, C_3, \cdots, C_n, \cdots$$

of $\mathfrak{M}$ (a *minimizing sequence*) such that

$$\lim_{n \to \infty} J_{C_n} = \mu.$$

The first question that arises here is the question of the existence of a limit curve of the sequence (Σ). Hilbert established some conditions for which such a limit curve exists.

Suppose a limit curve (call it $\bar{C}$) exists and belongs to $\mathfrak{M}$. If the equation

$$J_{\bar{C}} = \lim_{n \to \infty} J_{C_n}$$

holds, then

$$J_{\bar{C}} = \mu. \tag{2}$$

This means that the curve $\bar{C}$ furnishes an absolute minimum.

Equation (2) certainly holds if J_C is a continuous functional; that is, if the inequality

$$|J_C - J_{\bar{C}}| < \epsilon$$

is satisfied for every curve $C \in \mathfrak{M}$ in some ϵ-neighborhood of the

curve $\bar{C}$. However, J_C is not, in general, a continuous functional. In particular, as Lebesgue first noted, continuity is not at all necessary for the proof of equation (2) but lower semicontinuity is sufficient.*

Now suppose that the functional J_C is lower semicontinuous and that (Σ) is a minimizing sequence and also that $\bar{C}$ is its limit curve belonging to $\mathfrak{M}$. Then the definition of μ provides that the inequality

$$\mu \leq J_{\bar{C}} \tag{3}$$

holds and the lower semicontinuity of J_C insures that the inequality

$$J_{C_n} \geq J_{\bar{C}} - \epsilon$$

holds if C_n lies in a sufficiently small neighborhood of the curve $\bar{C}$. Because

$$\lim_{n \to \infty} J_{C_n} = \mu,$$

for $\epsilon' > 0$ we may write the inequality

$$\mu \geq J_{\bar{C}} - \epsilon'. \tag{4}$$

Relations (4) and (3) show that

$$J_{\bar{C}} = \mu$$

which means that the curve $\bar{C}$ furnishes a minimum for the functional J_C.

Thus a direct method consists of three steps:

A. *The construction of a minimizing sequence,*

---

*Let us recall the definition of semicontinuity for ordinary functions. The inequality

$$|f(x + h) - f(x)| < \epsilon$$

that appears in the definition of continuity, is equivalent to the two inequalities:

$$f(x + h) > f(x) - \epsilon, \qquad f(x + h) < f(x) + \epsilon.$$

If, for arbitrary $\epsilon > 0$, we can find a $\delta = \delta(\epsilon)$ such that for $|h| < \delta$ the first of these inequalities is satisfied, the function $f(x)$ is said to be *lower semicontinuous*. Similarly, the second inequality defines the condition for *upper semicontinuity*.

B. *Proof of the existence of a limit curve for this sequence,*

C. *Proof of the semicontinuity of the functional at the limit curve.*

28. THE ABSOLUTE MINIMUM FOR A FUNCTIONAL IN ORDINARY FORM

We shall consider the problem of an absolute minimum of the functional

$$J[y] = \phi[y(a)] + \int_a^b f(x, y, y')\, dx \tag{1}$$

for a fixed right-hand end point

$$y(b) = b_1 \tag{2}$$

with no condition imposed on the left-hand end point* and possibly some condition of the form

$$\int_a^b g(x, y)\, dx = l. \tag{3}$$

Because our applications of the construction are based on the present theory of functions of a real variable and assume integrability in the sense of Lebesgue, we shall consider the functional (1) in a broader collection than earlier: in the class $\mathfrak{A}$ of all absolutely continuous functions $y = y(x)$ that satisfy conditions (2) and (3). Here the curves $y = y(x)$ must lie in a closed region G of the xy-plane. We assume that the functions $f(x, y, z)$ and $g(x, y)$ are continuous in all their arguments for (x, y) in G and arbitrary finite z and that the function $\phi(y)$ is continuous and has a finite lower bound on the intersection of the region G with the straight line $x = a$. In all that follows we shall suppose that $\mathfrak{A}$ is an infinite set.

In addition to these assumptions we use two more; we formulate

---

*The construction introduced here can be used in the case of other boundary conditions; if both end points are fixed the problem is simplified because we can omit the term $\phi[y(a)]$ from the expression (1).

the first of these: There exist constants $\alpha > 0$, $p > 1$, β, for which

$$f(x, y, z) \geq \alpha|z|^p + \beta \tag{4}$$

for any z and any point $(x, y) \in G$.*

The assumption concerning the function $\phi(y)$ and condition (4) for an arbitrary function $y(x) \in \mathfrak{A}$ provide that

$$J[y] \geq \beta(b - a) + \gamma + \alpha \int_a^b |y'(x)|^p \, dx \tag{5}$$

where γ is the lower bound of $\phi(y)$. Hence

$$\mu = \inf_{y \text{ in } \mathfrak{A}} J[y] \geq \beta(b - a) + \gamma.$$

This insures the existence of a minimizing sequence which we shall designate as $\{y_n(x)\}$. It is evident that we may select $\{y_n(x)\}$ so that

$$J[y_n] \leq J[y_1] = A, \qquad (n = 1, 2, 3, \cdots).$$

In this case relation (5) provides that

$$\alpha \int_a^b |y_n'(x)|^p \, dx \leq A - \beta(b - a) - \gamma$$

so that there exists a constant B such that

$$\int_a^b |y_n'(x)|^p \, dx \leq B^p, \qquad (n = 1, 2, 3, \cdots). \tag{6}$$

Consider the following obvious relation

$$|y_n(x'') - y_n(x')| = \left| \int_{x'}^{x''} y_n'(x) \, dx \right| \leq \int_{x'}^{x''} |y_n'(x)| \, dx,$$
$$(a \leq x' < x'' \leq b).$$

---

*We can make condition (4) more general: There exist a constant β and a function $\psi(z)$ which approaches infinity as $|z| \to \infty$ for which

$$f(x, y, z) \geq |z|\psi(z) + \beta$$

for any z and for any point $(x, y) \in G$. This generalization was recently obtained by A. V. Pogorelov. A less general result was first used by Nagumo in a paper, "Über die gleichmässige Summierbarkeit und ihre Anwendung auf ein Variationsproblem," *Japanese Journal of Mathematics*, Vol. 6, 1930, pp. 173–182.

This and Hölder's* inequality imply that

$$|y_n(x'') - y_n(x')| \leq |x'' - x'|^{1/q} \left\{ \int_{x'}^{x''} |y_n'(x)|^p \, dx \right\}^{1/p}, \qquad (a \leq x' \leq x'' \leq b) \quad (7)$$

where q is defined by the equation

$$\frac{1}{p} + \frac{1}{q} = 1. \qquad (8)$$

The inequality (7) shows that

$$|y_n(x) - b_1| \leq (b - a)^{1/q} B$$

and

$$|y_n(x'') - y_n(x')| \leq B|x'' - x'|^{1/q}.$$

This means that the sequence $\{y_n(x)\}$ is uniformly bounded and equicontinuous. A theorem of Arzela implies that $\{y_n(x)\}$ contains a subsequence that converges uniformly to some function $\bar{y}(x)$. We designate this subsequence again by $\{y_n(x)\}$.

Now we show that the limit function $\bar{y}(x)$ is an admissible function. The fact that $\bar{y}(x)$ satisfies conditions (2) and (3) is obvious. It is not obvious that $\bar{y}(x)$ is an absolutely continuous function. To prove this, let us take an arbitrary finite system of nonoverlapping intervals $[x_i', x_i''] \subset [a, b]$. Then from relation (7)

$$\sum_{i=1}^{m} |y_n(x_i'') - y_n(x_i')| \leq \sum_{i=1}^{m} |x_i'' - x_i'|^{1/q} \left\{ \int_{x_i'}^{x_i''} |y_n'(x)|^p \, dx \right\}^{1/p}$$

and from Hölder's fundamental inequality for a sum, the relation

$$\sum_{i=1}^{m} |x_i'' - x_i'|^{1/q} \left\{ \int_{x_i'}^{x_i''} |y_n'(x)|^p \, dx \right\}^{1/p}$$

$$\leq \left\{ \sum_{i=1}^{m} |x_i'' - x_i'|^{1/q} \left\{ \sum_{i=1}^{m} \int_{x}^{x_i''} |y_n'(x)|^p \, dx \right\}^{1/p} \right.$$

$$\leq B \left\{ \sum_{i=1}^{m} |x_i'' - x_i'| \right\}^{1/q}$$

*For Hölder's inequality, see N. I. Akhiezer, *Lekcii po teorii approksimacii* (Moscow and Leningrad, 1947), p. 11; translated into English, N. J. Achieser, *Theory of Approximation*, C. J. Hyman, tr. (Frederick Ungar Publishing Co., Inc., New York, 1956). See also G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities* (Cambridge University Press, Cambridge, England, 1954).

holds so that

$$\sum_{i=1}^{m} |y_n(x_i'') - y_n(x_i')| \leq B \left\{ \sum_{i=1}^{m} |x_i'' - x_i'| \right\}^{1/q}.$$

Taking the limit as $n \to \infty$, we have

$$\sum_{i=1}^{m} |\bar{y}(x_i'') - \bar{y}(x_i')| \leq B \left\{ \sum_{i=1}^{m} |x_i'' - x_i'| \right\}^{1/q}.$$

The right-hand quantity in braces is the sum of the lengths of the chosen intervals. Thus the left side approaches zero with the sum of the lengths of the intervals $[x_i', x_i'']$; in other words, $\bar{y}(x)$ is an absolutely continuous function.

29. SOME AUXILIARY CONSIDERATIONS

It follows from relations (6) and (7) of Section 28, that the inequalities

$$\left\{ \int_a^b |y_n'(x)|^p \, dx \right\}^{1/p} \leq B \tag{1}$$

and

$$|y_n(x+h) - y_n(x)| \leq h^{1/q} \left\{ \int_x^{x+h} |y_n'(x)|^p \, dx \right\}^{1/p}, \quad (a \leq x < x+h \leq b) \tag{2}$$

hold. From relation (2) it follows that

$$\left| \frac{y_n(x+h) - y_n(x)}{h} \right|^p \leq \frac{1}{h} \int_x^{x+h} |y_n'(x)|^p \, dx = \frac{1}{h} \int_0^h |y_n'(x+t)|^p \, dt.$$

Hence

$$\int_a^{b-h} \left| \frac{y_n(x+h) - y_n(x)}{h} \right|^p dx \leq \frac{1}{h} \int_a^{b-h} dx \int_0^h |y_n'(x+t)|^p \, dt = \frac{1}{h} \int_0^h dt \int_a^{b-h} |y_n'(x+t)|^p \, dx \leq B^p.$$

The uniform convergence of the sequence $\{y_n(x)\}$ to $\bar{y}(x)$ implies that

$$\int_a^{b-h} \left| \frac{\bar{y}(x+h) - \bar{y}(x)}{h} \right|^p dx \leq B^p \tag{3}$$

and so for arbitrary fixed c in the interval (a, b) and for sufficiently small $h > 0$

$$\int_a^c \left| \frac{\bar{y}(x+h) - \bar{y}(x)}{h} \right|^p dx \leq B^p.$$

Because the function $\bar{y}(x)$ is absolutely continuous the ratio

$$\frac{\bar{y}(x+h) - \bar{y}(x)}{h}$$

converges to $\bar{y}'(x)$ as $h \to 0$ for almost all x. Hence the well-known theorem of Fatou and relation (3) imply that the function $|\bar{y}'(x)|^p$ is Lebesgue integrable and

$$\int_a^c |\bar{y}'(x)|^p \, dx \leq B^p.$$

Furthermore

$$\int_a^b |\bar{y}'(x)|^p \, dx \leq B^p \tag{4}$$

because c was an arbitrary point of the interval (a, b). Inequalities (1) and (4) show that the functions $y_n'(x)$ as well as the function $\bar{y}'(x)$ belong to the so-called L^p space.

Now we show that the sequence $\{y_n'(x)\}$ *converges weakly* to $\bar{y}'(x)$. The notion of weak convergence is one of the fundamental and general notions of functional analysis. In this case it means that

$$\lim_{n \to \infty} \int_a^b \{y_n'(x) - \bar{y}'(x)\} \psi(x) \, dx = 0$$

for any function $\psi(x)$ that is measurable and satisfies the inequality

$$\int_a^b |\psi(x)|^q \, dx < \infty.$$

In other words, $\psi(x)$ lies in L^q [where q is defined by equation (8) of Section 28].

To prove that the sequence converges weakly, we must use the following fact* from the theory of approximation: For an arbitrary function $\psi(x)$ in L^q and an arbitrary $\epsilon > 0$ we can find a polynomial $P_N(x)$ of degree $N = N(\epsilon)$ such that

$$\int_a^b |\psi(x) - P_N(x)|^q \, dx < \epsilon^q.$$

Because relations (1) and (4) imply that

$$\left\{\int_a^b |y_n'(x) - \bar{y}'(x)|^p \, dx\right\}^{1/p} \leq 2B \tag{5}$$

Hölder's inequality provides that

$$\begin{aligned}\left|\int_a^b \{y_n'(x) - \bar{y}'(x)\}\{\psi(x) - P_N(x)\} \, dx\right| &\leq \left\{\int_a^b |y_n'(x) - \bar{y}'(x)|^p \, dx\right\}^{1/p} \left\{\int_a^b |\psi(x) - P_N(x)|^q \, dx\right\}^{1/q} \\ &\leq 2B\epsilon.\end{aligned}$$

Hence

$$\begin{aligned}\overline{\lim_{n\to\infty}} \left|\int_a^b \{y_n'(x) - \bar{y}'(x)\}\psi(x) \, dx\right| &\leq 2B\epsilon + \overline{\lim_{n\to\infty}} \left|\int_a^b \{y_n'(x) - \bar{y}'(x)\}P_N(x) \, dx\right|\end{aligned}$$

and it is sufficient to show that for an arbitrary polynomial $P(x)$

$$\lim_{n\to\infty} \int_a^b \{y_n'(x) - \bar{y}'(x)\}P(x) \, dx = 0. \tag{6}$$

Then

$$\begin{aligned}\int_a^b \{y_n'(x) - \bar{y}'(x)\}P(x) \, dx = \{y_n(x) - \bar{y}(x)\}P(x)\Big|_{x=a}^{x=b} \\ - \int_a^b \{y_n(x) - \bar{y}(x)\}P'(x) \, dx\end{aligned}$$

*See N. I. Akhiezer, *Lekcii po teorii approksimacii* (Moscow and Leningrad, 1947); available in English, N. J. Achieser, *Theory of Approximation*, C. J. Hyman, tr. (Frederick Ungar Publishing Co., Inc., New York, 1956).

and the proof of equation (6) follows from the uniform convergence of the sequence $\{y_n(x)\}$ to $\bar{y}(x)$.

30. A THEOREM ABOUT ABSOLUTE MINIMA

Let us return to our problem. We have shown that in the collection $\mathfrak{A}$ there exists a minimizing sequence $\{y_n(x)\}$ which converges uniformly to some function $\bar{y}(x)$; and we have also shown that the sequence of derivatives $\{y_n'(x)\}$ converges weakly to $\bar{y}'(x)$. To do this we used only condition (4) of Section 28. We now impose on the function $f(x, y, z)$ the second of the conditions mentioned in Section 28. This condition can be considered as a weak variant of an assumption about quasi-regularity for the functional; it is formulated as follows: $f(x, y, z)$ has a continuous derivative $f_z(x, y, z)$, and for any point (x, y) in G this derivative is a nondecreasing function of z, $(-\infty < z < \infty)$.

Making use of this property of $f(x, y, z)$ we shall prove the inequality

$$\mu = \lim_{n \to \infty} J[y_n] \geq J[\bar{y}] \tag{1}$$

from which it will follow that

$$J[\bar{y}] = \mu.$$

In other words, *the functional* $J[y]$, *for the given assumptions concerning the function* $f(x, y, z)$, *has an absolute minimum in the collection* $\mathfrak{A}$ *of all absolutely continuous functions that satisfy conditions* (2) *and* (3) *of Section* 28. Tonelli first stated this important theorem.

To prove relation (1), we designate by E_M the set of points of the interval $[a, b]$, at which

$$|\bar{y}'(x)| \leq M.$$

Consider the identity

$$f(x, y_n, y_n') - \beta = f(x, \bar{y}, \bar{y}') - \beta + \{f(x, y_n, y_n') - f(x, y_n, \bar{y}')\} + \{f(x, y_n, \bar{y}') - f(x, \bar{y}, \bar{y}')\}$$

where β is the constant of formula (4) of Section 28. Because $y_n(x)$ converges uniformly to $\bar{y}(x)$, we can find an $N(\epsilon)$ such that for

arbitrary $\epsilon > 0$ and for $n > N(\epsilon)$

$$|f(x, y_n, \bar{y}') - f(x, \bar{y}, \bar{y}')| < \epsilon$$

at every point of the set E_M. Hence the relation written above implies that for $n > N(\epsilon)$

$$\int_{E_M} \{f(x, y_n, y_n') - \beta\}\, dx \geq \int_{E_M} \{f(x, \bar{y}, \bar{y}') - \beta\}\, dx + \int_{E_M} \{f(x, y_n, y_n') - f(x, y_n, \bar{y}')\}\, dx - \epsilon(b - a). \quad (2)$$

On the other hand, the monotonicity of $f_z(x, y, z)$ insures that the following inequality holds:

$$f(x, y_n, y_n') - f(x, y_n, \bar{y}') \geq (y_n' - \bar{y}') f_z(x, y_n, \bar{y}').$$

Because $f(x, y, z) - \beta \geq 0$, inequality (2) implies that

$$\begin{aligned}\int_a^b \{f(x, y_n, y_n') - \beta\}\, dx &\geq \int_{E_M} \{f(x, \bar{y}, \bar{y}') - \beta\}\, dx \\ &+ \int_{E_M} (y_n' - \bar{y}') f_z(x, y_n, \bar{y}')\, dx - \epsilon(b - a) \\ &= \int_{E_M} \{f(x, \bar{y}, \bar{y}') - \beta\}\, dx \\ &+ \int_{E_M} (y_n' - \bar{y}')\{f_z(x, y_n, \bar{y}') - f_z(x, \bar{y}, \bar{y}')\}\, dx \\ &+ \int_{E_M} (y_n' - \bar{y}') f_z(x, \bar{y}, \bar{y}')\, dx - \epsilon(b - a). \quad (3)\end{aligned}$$

It is not difficult to show that the second and third integrals on the right approach zero as $n \to \infty$. If we set

$$\psi(x) = \begin{cases} f_z(x, \bar{y}, \bar{y}') & x \in E_M \\ 0 & x \bar{\in} E_M \end{cases}$$

then $\psi(x) \in L^q$ and then

$$\lim_{n \to \infty} \int_{E_M} (y_n' - \bar{y}') f_z(x, \bar{y}, \bar{y}')\, dx = \lim_{n \to \infty} \int_a^b (y_n' - \bar{y}')\psi(x)\, dx = 0$$

because $\{y_n'(x)\}$ converges weakly to $\bar{y}'(x)$. On the other hand,

with the aid of Hölder's inequality and formula (5) of Section 29, we obtain

$$\left| \int_{E_M} (y_n' - \bar{y}')\{f_z(x, y_n, \bar{y}') - f_z(x, \bar{y}, \bar{y}')\}\, dx \right| \leq 2B \left\{ \int_{E_M} |f_z(x, y_n, \bar{y}') - f_z(x, \bar{y}, \bar{y}')|^q\, dx \right\}^{1/q}$$

The difference

$$f_z(x, y_n, \bar{y}') - f_z(x, \bar{y}, \bar{y}')$$

converges uniformly to zero on the set E_M as $n \to \infty$, thus

$$\lim_{n \to \infty} \int_{E_M} (y_n' - \bar{y}')\{f_z(x, y_n, \bar{y}') - f_z(x, \bar{y}, \bar{y}')\}\, dx = 0.$$

After passage to the limit, inequality (3) becomes

$$\lim_{n \to \infty} \int_a^b \{f(x, y_n, y_n') - \beta\}\, dx \geq \int_{E_M} \{f(x, \bar{y}, \bar{y}') - \beta\}\, dx - \epsilon(b - a)$$

or

$$\mu \geq \phi[\bar{y}(a)] + \int_{E_M} \{f(x, \bar{y}, \bar{y}') - \beta\}\, dx + (\beta - \epsilon)(b - a).$$

Hence, by increasing M without bound we obtain

$$\mu \geq \phi[\bar{y}(a)] + \int_a^b \{f(x, \bar{y}, \bar{y}') - \beta\}\, dx + (\beta - \epsilon)(b - a)$$

or

$$J[\bar{y}] \leq \mu + \epsilon(b - a).$$

Because $\epsilon > 0$ is arbitrary, both inequality (1) and Tonelli's theorem about the minimum have been proved.

31. THE EULER EQUATION

Suppose that the function $\bar{y}(x)$ furnishes an absolute minimum for the functional

$$J[y] = \phi[y(a)] + \int_a^b f(x, y, y')\, dx$$

for the condition

$$y(b) = b_1$$

and relations

$$\int_a^b g_i(x, y)\, dx = l_i, \qquad (i = 1, 2, 3, \cdots, k)$$

which we now describe completely. We have only proved the existence of this function in a class of absolutely continuous functions; this is all we asserted for functions f, g_i, ϕ given above.

Now we suppose that the functions f, g_i, ϕ all have continuous derivatives with respect to y. In such a case, if the curve

$$y = \bar{y}(x)$$

with the possible exception of its end points lies in G and if the functions $\bar{f}_y$ and $\bar{f}^2_{y'}$ are summable, then we can assert that there exist constants

$$C, \lambda_i, \qquad \left(\sum_0^k \lambda_i^2 \neq 0\right)$$

for which almost everywhere in $[a, b]$

$$\lambda_0 \bar{f}_{y'} - \int_a^x [\lambda_0 \bar{f}_y - \lambda_1 \bar{g}_{1y} - \cdots - \lambda_k \bar{g}_{ky}]\, dx = C. \tag{1}$$

If we vary the curve $y = \bar{y}(x)$ as we did in Section 25, we find that for certain constants

$$C, \lambda_i, \qquad \left(\sum_0^k \lambda_i^2 \neq 0\right),$$

$$\int_a^b \left\{\lambda_0 \bar{f}_{y'} - \int_a^x [\lambda_0 \bar{f}_y - \lambda_1 \bar{g}_{1y} - \cdots - \lambda_k \bar{g}_{ky}]\, dx - C\right\}^2 = 0.$$

It follows that the expression in braces vanishes almost everywhere in the interval $[a, b]$. This gives us the Euler equation (1).

The most important case is that in which the assumption concerning the monotonicity of f_z is replaced with the stricter assumption that for arbitrary finite z and arbitrary points $(x, y) \in G$, the continuous second derivative f_{zz} exists and

$$f_{zz}(x, y, z) > 0. \tag{2}$$

If we put aside the exceptional case, $\lambda_0 = 0$ in equation (1), then we can set $\lambda_0 = 1$. Then let us consider the equation

$$f_{y'}(x, \bar{y}, z) - \int_a^x [f_y(x, \bar{y}, \bar{y}') - \lambda_1 \bar{g}_{1y} - \cdots - \lambda_k \bar{g}_{ky}]\, dx = C.$$

A theorem about implicit functions which applies here because of condition (2) implies that this equation defines uniquely a function $z = z(x)$ continuous in whole interval $[a, b]$ and that the function $z = z(x)$ is equal to $\bar{y}'(x)$ almost everywhere in this interval. Hence, under condition (2) the absolutely continuous function $\bar{y}(x)$ is a smooth function, and consequently satisfies the Euler equation in differentiated form:

$$\frac{d}{dx} f_{y'} - [f_y - \lambda_1 g_{1y} - \cdots - \lambda_k g_{ky}] = 0. \tag{3}$$

To illustrate these general results we consider the linear equation

$$-\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + q(x)y = g(x) \tag{4}$$

in which $p(x) > 0$, $q(x) \geq 0$, and $g(x)$ is a known continuous function in the finite interval $[a, b]$. Suppose we wish to find a solution of this equation with boundary conditions

$$y(a) = a_1, \qquad y(b) = b_1. \tag{5}$$

Equation (4) is the Euler equation for the quadratic functional

$$J[y] = \int_a^b \{p(x)y'^2 + q(x)y^2 - 2g(x)y\}\, dx$$

such that

$$f(x, y, z) = p(x)z^2 + q(x)y^2 - 2g(x)y.$$

First of all we show that to find a minimum for the functional $J[y]$ we can restrict ourselves to those curves which lie in a rectangle

$$a \leq x \leq b, \qquad -m \leq y \leq m. \tag{6}$$

For this purpose take for example some linear function $y_0(x)$ that satisfies conditions (5) and set

$$J[y_0] = A.$$

To find a minimum for the functional $J[y]$, we can evidently restrict ourselves to the consideration of all admissible functions $y = y(x)$, for which

$$J[y] \leq A.$$

Because $q(x) \geq 0$, it follows that

$$\int_a^b p(x)y'^2\,dx \leq A + 2\int_a^b g(x)y\,dx.$$

Hence*

$$p_{\min}\int_a^b y'^2\,dx \leq A + 2|y|_{\max}\int_a^b |g(x)|\,dx$$

and this implies that

$$\int_a^b y'^2\,dx \leq A_1 + A_2|y|_{\max} \tag{7}$$

where A_1 and A_2 are well defined constants. On the other hand, the equation

$$y(x) = a_1 + \int_a^x y'(x)\,dx$$

implies that

$$|y|_{\max} \leq |a_1| + \int_a^b |y'(x)|\,dx \leq |a_1| + \sqrt{b-a}\sqrt{\int_a^b y'^2\,dx}. \tag{8}$$

Comparing relations (8) and (7) yields

$$|y|_{\max} \leq |a_1| + \sqrt{b-a}\sqrt{A_1 + A_2|y|_{\max}}$$

from which it follows that

$$|y|_{\max} \leq m$$

in which m depends only on $b - a$, $|a_1|$, A_1, A_2.

Thus the curves $y = y(x)$, along which we consider the functional $J[y]$, must lie in a rectangle (6) which takes the place of the region G. In this rectangle

$$f(x, y, z) \geq \alpha z^2 + \beta$$

---

*We use the notation

$$\phi_{\min} = \min_{a \leq x \leq b} \phi(x), \qquad \phi_{\max} = \max_{a \leq x \leq b} \phi(x).$$

in which

$$\alpha = p_{\min} \qquad \text{and} \qquad \beta = -2m|g|_{\max}$$

that is, condition (4) of Section 28 is satisfied at each point of the region (6) for arbitrary finite z if we take $p = 2$.

The condition of monotonicity of $f(x, y, z)$ is satisfied because

$$f_z(x, y, z) = 2p(x)z.$$

A stronger condition

$$f_{zz}(x, y, z) = 2p(x) \geq 2p_{\min} > 0$$

is also satisfied; hence a curve $y = \bar{y}(x)$ exists that supplies the functional $J[y]$ with an absolute minimum, is also smooth and satisfies equation (4). Thus we have shown that a solution of equation (4) under conditions (5) exists.

Proof of the uniqueness of the solution can be shown very quickly. If there were two solutions $y_1(x), y_2(x)$ then their difference $u(x) = y_1(x) - y_2(x)$ would satisfy the homogeneous equation

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} - q(x)u = 0 \tag{9}$$

and the homogeneous conditions

$$u(a) = u(b) = 0.$$

If we multiply the expression (9) by u and integrate by parts, we find that

$$\int_a^b \{pu'^2 + qu^2\}\, dx = 0$$

which is possible only if $u \equiv 0$ because of the assumptions about the functions $p(x)$ and $q(x)$.

32. THE CONSTRUCTION OF A MINIMIZING SEQUENCE

V. Ritz devised an important method for constructing a minimizing sequence; it has been applied with success to various engineering problems. This and the similar method of B. G. Galerki

are found in detail in a series of textbooks and special articles.* We shall restrict ourselves here to an explanation of the Ritz method for the simplest problem of an absolute minimum of the functional

$$J[y] = \int_a^b f(x, y, y')\, dx$$

subject only to the conditions

$$y(a) = a_1, \qquad y(b) = b_1. \tag{1}$$

We shall suppose that $f(x, y, z)$ is continuous in all its arguments in some closed region G of the xy-plane for finite values of z and satisfies the inequality

$$f(x, y, z) \geq \alpha|z|^p + \beta \tag{2}$$

for certain constants $\alpha > 0$, β, $p > 1$, and also the other conditions mentioned above are fulfilled; these conditions taken all together guarantee the existence of a function $\bar{y}(x)$ which furnishes $J[y]$ with an absolute minimum

$$\mu = J[\bar{y}].$$

For this purpose we consider the case when $y = \bar{y}(x)$, $(a \leq x \leq b)$ is a smooth curve, lying in the region G with the possible exception of its end points.

We take a sequence of continuously differentiable functions $\{\phi_k(x)\}_0^\infty$, $(a \leq x \leq b)$, that satisfy the following five conditions:

A. $\phi_0(a) = a_1, \quad \phi_0(b) = b_1.$

B. $\phi_k(a) = \phi_k(b) = 0, \quad (k = 1, 2, \cdots).$

C. The curve $y = \phi_0(x)$, $(a \leq x \leq b)$ with the possible exception of its end points lies in the region G.

D. For arbitrary n the functions $\phi_1'(x), \phi_2'(x), \cdots, \phi_n'(x)$ are linearly independent.

E. For every continuous function $F(x)$ for arbitrary $\epsilon > 0$ there is a "polynomial" $[n = n(\epsilon)]$

$$F_n(x) = A_1\phi_1'(x) + A_2\phi_2'(x) + \cdots + A_n\phi_n'(x)$$

*See L. V. Kantorovich and V. I. Krylov, *Pribiženny metodi vysshevo analiza* (Gostehizdat, 1949); available in English as *Approximate Methods of Higher Analysis*, C. D. Brewster, tr. (Interscience Publishers, Inc., New York, 1958). Also see S. G. Mihlin, *Problema minima kvadraticnogo funkcionala* (Gostehizdat, 1942).

that satisfies the inequality

$$|F(x) - F_n(x)| \leq \epsilon.$$

Conditions A, B, D, E will certainly be satisfied if

$$\phi_0(x) = a_1 + \frac{b_1 - a_1}{b - a}(x - a)$$

and

$$\phi_k(x) = (x - a)^k(x - b), \qquad (k = 1, 2, \cdots)$$

or

$$\phi_k(x) = \sin \frac{k\pi(x - a)}{b - a}, \qquad (k = 1, 2, \cdots).$$

We turn now to our calculus of variations problem. Let us take the "polynomial"

$$y_n(x) = \phi_0(x) + C_1\phi_1(x) + C_2\phi_2(x) + \cdots + C_n\phi_n(x), \quad (3)$$

which satisfies the boundary conditions A for arbitrary constants C_i, $(i = 1, 2, \cdots, n)$; we choose these constants so that the curve

$$y = y_n(x) \quad (4)$$

lies in the region G. Thus the functional $J[y]$ has meaning for $y = y_n(x)$ and is transformed by it into a continuous function of C_i, $(i = 1, 2, \cdots, n)$:

$$J[y_n] = \Phi(C_1, C_2, \cdots, C_n). \quad (5)$$

We are seeking numbers C_i such that $\Phi(C_1, C_2, \cdots, C_n)$ assumes a least value. Thus we have an elementary problem of finding a minimum, and the first question that arises here is whether or not this problem can be solved.

We shall show that the answer to this question is affirmative. For this purpose we select constants C_i, $(i = 1, 2, \cdots, n)$ for which the curve (4) lies in the region G, and designate the corresponding value of the function (5) by M. Since we seek a minimum for the function (5) we can restrict ourselves to consider only those polynomials (3) for which

$$\Phi(C_1, C_2, \cdots, C_n) \leq M. \quad (6)$$

On the basis of inequality (2) it follows from inequality (6) that

$$\int_a^b \left| \phi_0'(x) + \sum_{i=1}^{n} C_i \phi_i'(x) \right|^p dx \leq \frac{M - \beta(b-a)}{\alpha} = M_1$$

thus

$$\left\{ \int_a^b \left| \sum_{i=1}^{n} C_i \phi_i'(x) \right|^p dx \right\}^{1/p} \leq M_1^{1/p} + \left\{ \int_a^b |\phi_0'(x)|^p \, dx \right\}^{1/p} = M_2. \quad (7)$$

The left side can be put in the form

$$\sqrt{C_1^2 + C_2^2 + \cdots + C_n^2} \left\{ \int_a^b \left| \sum_{i=1}^{n} K_i \phi_i'(x) \right|^p dx \right\}^{1/p} \quad (8)$$

in which

$$K_i = \frac{C_i}{\sqrt{C_1^2 + C_2^2 + \cdots + C_n^2}}, \qquad (i = 1, 2, \cdots, n)$$

which means that

$$K_1^2 + K_2^2 + \cdots + K_n^2 = 1. \quad (9)$$

The function

$$\left\{ \int_a^b \left| \sum_{i=1}^{n} K_i \phi_i'(x) \right|^p dx \right\}^{1/p}$$

is continuous on the unit sphere (9) and hence, according to a theorem of Weierstrass, assumes a minimum value δ on it; this minimum value is evidently greater than zero, because the functions $\phi_1'(x), \phi_2'(x), \cdots, \phi_n'(x)$ are linearly independent. Consequently the function (8) has a value $\geq \delta \sqrt{C_1^2 + C_2^2 + \cdots + C_n^2}$ for arbitrary $C_1, C_2, \cdots, C_n$. But then it follows from inequality (7) that

$$\sqrt{C_1^2 + C_2^2 + \cdots + C_n^2} \leq \frac{M_2}{\delta}. \quad (10)$$

Thus, to satisfy inequality (6) it is necessary to satisfy inequality (10) and consequently we must seek a minimum of the function (5)

in a bounded closed region (10) of n-dimensional space; a continuous function (5) has a minimum in such a region. Our elementary minimum problem thus has a solution for arbitrary n.

We denote the sequence of minimal polynomials obtained in this manner by $\{\bar{y}_n(x)\}_1^\infty$. In so far as each polynomial $y_n(x)$ is a special case of polynomial $y_{n+1}(x)$, the following inequality holds:

$$J[\bar{y}_{n+1}] \leq J[\bar{y}_n].$$

It follows from this that the quantities

$$\mu_n = J[\bar{y}_n]$$

are monotonic decreasing as $n \to \infty$ and hence converge to some value μ^* concerning which we can only assert in advance that

$$\mu^* \geq \mu.$$

Using property E, we show that the inequality sign in this relation can be omitted; by showing this we prove that the sequence of minimal polynomials $\{\bar{y}_n(x)\}_1^\infty$ is a minimizing sequence for our functional.

For the proof we use an arbitrary $\epsilon > 0$ and construct the polynomial

$$S_n'(x) = A_1\phi_1'(x) + A_2\phi_2'(x) + \cdots + A_n\phi_n'(x)$$

for which in the whole interval $[a, b]$

$$|\bar{y}'(x) - \phi_0'(x) - S_n'(x)| < \epsilon.$$

By integrating this inequality from a to x we find that

$$|\bar{y}(x) - \phi_0(x) - S_n(x)| < \epsilon(b - a).$$

Now take the polynomial

$$\tilde{y}_n(x) = \phi_0(x) + A_1\phi_1(x) + A_2\phi_2(x) + \cdots + A_n\phi_n(x)$$

and consider the difference

$$J[\tilde{y}_n] - J[\bar{y}] = \int_a^b \{f(x, \tilde{y}_n, \tilde{y}_n') - f(x, \bar{y}, \bar{y}')\}\, dx. \qquad (11)$$

The continuity of $f(x, y, z)$ and the inequalities

$$\left.\begin{aligned} |\tilde{y}_n(x) - \bar{y}(x)| &< \epsilon(b - a) \\ |\tilde{y}_n'(x) - \bar{y}'(x)| &< \epsilon \end{aligned}\right\} \qquad (a \leq x \leq b)$$

where $\epsilon \to 0$ as $n \to \infty$ imply that the right side of equation (11) approaches zero as $n \to \infty$. Hence for arbitrary $\eta > 0$ we can find an $N(\eta)$ such that for $n > N(\eta)$

$$0 \leq J[\tilde{y}_n] - J[\bar{y}] < \eta$$

and so

$$J[\tilde{y}_n] < \mu + \eta.$$

But the definition of the minimal polynomials $\bar{y}_n(x)$ implies that

$$J[\bar{y}_n] \leq J[\tilde{y}_n].$$

Consequently

$$\mu^* \leq J[\bar{y}_n] < \mu + \eta$$

from which it follows that

$$\mu^* = \mu.$$

33. THE ABSOLUTE MINIMUM PROBLEM FOR A FUNCTIONAL IN PARAMETRIC FORM

Let us consider the functional in parametric form:

$$J_C = \int_C F(x, y, \dot{x}, \dot{y})\, dt. \tag{1}$$

We suppose that the continuous function F has continuous second derivatives and is homogeneous of the first degree with respect to $\dot{x}$ and $\dot{y}$. Consequences of the homogeneity of F are stated in Section 10. Curves C for which we consider the functional J_C are assumed to be rectifiable. We may suppose that the integral exists as a Lebesgue integral. If the curve

$$x = x(t), \qquad y = y(t)$$

is rectifiable we can take the arc length s as the parameter. The

equation of the curve will be

$$x = \phi(s), \qquad y = \psi(s).$$

Now

$$|\phi(s + \Delta s) - \phi(s)| \leq |\Delta s|,$$
$$|\psi(s + \Delta s) - \psi(s)| \leq |\Delta s|$$

which shows that $\phi'(s)$ and $\psi'(s)$ exist almost everywhere and do not exceed unity. Hence the integral

$$J_C = \int_C F(x, y, \dot{x}, \dot{y})\, dt = \int_C F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) ds$$

exists in the sense of Lebesgue. Our problem is to prove the following theorem.

THEOREM: *If a functional* (1) *is quasi-regular in a bounded closed region G, that is, for* (x, y) *in G and arbitrary* θ

$$F_1(x, y, \cos\theta, \sin\theta) \geq 0, \tag{2}$$

and if

$$F(x, y, \cos\theta, \sin\theta) \geq \omega > 0 \tag{3}$$

then in every collection of rectifiable curves lying in G, there exists at least one curve that furnishes the functional with an absolute minimum.

The proof of this theorem falls into three parts. In the present section we present an outline of the proof and the formulation of necessary preliminary propositions. In the following three sections we shall prove these propositions and, at the same time, the theorem.

For every admissible curve C whose length we designate by l the following inequality holds

$$J_C \geq \omega l > 0. \tag{4}$$

Hence the set of values of the functional J_C has a lower bound; the existence of a minimizing sequence $\{C_n\}_1^\infty$ follows from this and we can assume that as $n \to \infty$, the monotonic decreasing integral J_{C_n} approaches

$$\mu = \inf J_C.$$

Inequality (4) provides that the length l_n of the curve C_n satisfies the inequality

$$l_n \leq \frac{1}{\omega} J_{C_n}$$

from which it follows that

$$l_n \leq \frac{1}{\omega} J_{C_1} = \Lambda, \qquad (n = 1, 2, \cdots).$$

Since we are seeking a minimum, we can restrict the proof to those curves whose length does not exceed Λ.

In accordance with the plan described in Section 27, we must show that the sequence $\{C_n\}_1^\infty$ has a limit curve. To accomplish this we shall use Hilbert's theorem.

HILBERT'S THEOREM: *An infinite collection of rectifiable curves whose lengths are bounded and that lie in a bounded closed region has at least one limit curve which is rectifiable and which lies in the same region.* This theorem establishes that the minimizing sequence $\{C_n\}_1^\infty$ has a limit curve $\bar{C}$. Note that only condition (3) is used for the proof of this theorem.

In order to show that the curve $\bar{C}$ furnishes J_C with an absolute minimum we must use condition (2) which states that the functional J_C is quasi-regular. We need the following theorem of Tonelli for this demonstration.

TONELLI'S THEOREM: *Every quasi-regular functional is lower semi-continuous in a collection of curves whose lengths are bounded.*

34. PROOF OF HILBERT'S THEOREM

Suppose we have an infinite collection $\mathfrak{M}$ of curves C, lying in a bounded closed region, and suppose some number Λ is an upper bound for the lengths of these curves. For any curve C in $\mathfrak{M}$, take as parameter

$$t = \frac{s}{l}$$

where l is the length of the whole curve, and s is arc length measured

from an initial point of the curve. The equation of the curve C will be

$$x = x(t), \qquad y = y(t), \qquad (0 \leq t \leq 1).$$

Thus, two families $\{x(t)\}$ and $\{y(t)\}$ of uniformly bounded functions belong to the collection $\mathfrak{M}$. Furthermore,

$$|x(t + \Delta t) - x(t)| \leq |\Delta s| = l|\Delta t| \leq \Lambda|\Delta t|,$$

$$|y(t + \Delta t) - y(t)| \leq |\Delta s| = l|\Delta t| \leq \Lambda|\Delta t|$$

and so all the functions $x(t)$, $y(t)$ are uniformly continuous. Thus, from a theorem of Arzela, we have a pair of limit functions

$$x = \phi(t), \qquad y = \psi(t).$$

From this we have the inequalities

$$|\phi(t + \Delta t) - \phi(t)| \leq \Lambda|\Delta t|,$$

$$|\psi(t + \Delta t) - \psi(t)| \leq \Lambda|\Delta t|$$

and it follows that the derivatives $\phi'(t)$, $\psi'(t)$ exist almost everywhere and do not exceed Λ in absolute value. This implies that the limit curve

$$x = \phi(t), \qquad y = \psi(t), \qquad (0 \leq t \leq 1)$$

is rectifiable, and completes the proof of Hilbert's theorem.

35. AN AUXILIARY PROPOSITION

We shall prove the following lemma before giving a proof of Tonelli's theorem.

LEMMA (ROUSSEL): *The line integral*

$$I_C = \int_C \left\{ P(x, y) \frac{dx}{ds} + Q(x, y) \frac{dy}{ds} \right\} ds$$

*in which P and Q are continuous functions, is a continuous function in the collection of curves C, whose lengths are bounded.**

*A. Roussel, "Recherches sur le calcul des variations," *Journ. de math.*, Vol. 5, 1926, pp. 395–462.

PROOF: Suppose C_0 is a curve of the collection, and D is a bounded region containing C_0. Given a number $\alpha > 0$, the continuity of P and Q in D provides that there exists a $\delta > 0$, such that the oscillation of neither function P nor Q exceeds α in any circle of radius δ with center in D. The curve C_0 can be covered by a finite number of these circles and consequently the covered curve C_0 can be divided into $n = n(\alpha)$ arcs C_0^i, $(i = 1, 2, \cdots, n)$ enumerated relative to increasing arc length so that each of the arcs C_0^i lies wholly within one of the designated circles. If necessary, we may add a finite number of circles of radius δ and assume that each of the arcs C_0^i corresponds to the circle containing it. These circles will also be numbered the same as the arcs contained in them. Designate the circles by K^i, $(i = 1, 2, \cdots, n)$.

Suppose that Λ is the upper bound of the lengths of the curves C, and that M is the upper bound of $|P|$, $|Q|$ in the region D. Given an arbitrary $\epsilon > 0$ we restrict $\alpha > 0$ so that it satisfies the inequality $\alpha \leq \epsilon/8\Lambda$. Having α, we determine $\delta > 0$ as shown above, and then $n = n(\alpha)$. Next we consider a $\rho > 0$ that satisfies the inequality

$$\rho \leq \frac{\epsilon}{8(n+1)M}$$

and has the property that the circle of radius ρ with center at an arbitrary point of the arc C_0^i lies in the circle K^i, $(i = 1, 2, \cdots, n)$.

Then we take any curve C of our collection lying in a strong ρ-neighborhood* of the curve C_0, and show that

$$|I_C - I_{C_0}| \leq \epsilon.$$

This will serve to prove the Roussel lemma.

Recall that, according to the definition set down in Section 10, a curve C belongs to a ρ-neighborhood of the curve C_0 if there exists a parametrization

$$C: \quad x = x(t), \qquad y = y(t)$$

$$C_0: \quad x = x_0(t), \qquad y = y_0(t), \qquad (t_0 \leq t \leq T)$$

*Here and in what follows we mean a strong neighborhood.

such that

$$\sqrt{[x(t) - x_0(t)]^2 + [y(t) - y_0(t)]^2} \leq \rho, \qquad (t_0 \leq t \leq T).$$

Suppose that a subdivision, $t_0 < t_1 < t_2 < \cdots < t_{n-1} < T$, of the interval $[t_0, T]$ corresponds to the subdivision of the curve C_0 into arcs C_0^i. It generates a subdivision of the curve C into subarcs C^i; each subarc C^i lies wholly in the circle K^i. Consider the integral

$$I_C^* = \sum_{j=1}^{n} \int_{C^j} \{P_j x' + Q_j y'\}\, ds$$

where P_j and Q_j are the values of the functions P and Q at the center of the circle K^j. Then

$$|I_C - I_C^*| \leq \sum_{j=1}^{n} \int_{C^j} \{|P - P_j| + |Q - Q_j|\}\, ds$$

$$\leq 2\alpha \sum_{j=1}^{n} \int_{C^j} ds \leq 2\alpha\Lambda \quad (1)$$

and analogously

$$|I_{C_0} - I_{C_0}^*| \leq 2\Lambda\alpha. \quad (2)$$

On the other hand

$$\oint \{P_j x' + Q_j y'\}\, ds = 0.$$

Hence if we add and compute the integrals along the transverse segments σ_j', σ_j'' as in Figure 5 we reach the inequality

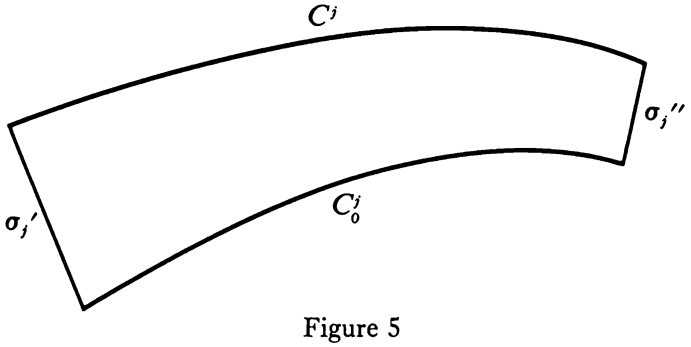

Figure 5

$$|I_C^* - I_{C_0}^*| \leq \sum_0^n 4M\rho = 4(n+1)M\rho. \tag{3}$$

Taking into account relations (1), (2), and (3) we find that

$$|I_C - I_{C_0}| \leq |I_C - I_C^*| + |I_C^* - I_{C_0}^*| + |I_{C_0}^* - I_{C_0}|$$
$$\leq 4\Lambda\alpha + 4(n+1)M\rho \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which was to be proved.

36. PROOF OF TONELLI'S THEOREM

Suppose

$$J_C = \int_C F(x, y, x', y')\, ds$$

is a given quasi-regular functional and suppose that C_0 is an admissible curve. We inscribe in the curve C_0 a broken line Γ_0 with m sides Γ_0^i, $(i = 1, 2, \cdots, m)$; we denote by θ_i the angle that the sides make with the axis Ox. By C_0^i, $(i = 1, 2, \cdots, m)$ we designate the arcs into which the curve is divided by the vertices of the broken line Γ_0. If some curve C lies in the neighborhood of C_0 then the subdivision of the curve into arcs C_0^i generates a subdivision of the curve C into arcs C^i, $(i = 1, 2, \cdots, m)$. Thus we can construct an auxiliary functional J_C^* that depends on the broken line Γ_0:

$$J_C^* = \sum_{i=1}^m \int_{C^i} \left\{ F_p(x, y, \alpha_i, \beta_i) \frac{dx}{ds} + F_q(x, y, \alpha_i, \beta_i) \frac{dy}{ds} \right\} ds$$

in which

$$\alpha_i = \cos\theta_i, \qquad \beta_i = \sin\theta_i.$$

From the homogeneity of F in its second pair of arguments on the segment Γ_0^i we have the equation

$$F_p(x, y, \alpha_i, \beta_i) \frac{dx}{ds} + F_q(x, y, \alpha_i, \beta_i) \frac{dy}{ds} = F(x, y, \alpha_i, \beta_i).$$

Hence

$$J_{\Gamma_0} = J^*_{\Gamma_0}. \tag{1}$$

But J^*_C is the sum of integrals of the type that appear in Roussel's lemma. Hence, for a fixed broken line Γ_0, we can determine a $\rho = \rho(\epsilon)$ such that, if the curve C of our collection lies in the ρ-neighborhood of the curve C_0, then

$$|J^*_C - J^*_{C_0}| \leq \frac{\epsilon}{3} \tag{2}$$

whatever value $\epsilon > 0$ has.

Let us suppose for a moment that the broken line can be chosen so that

$$|J_{C_0} - J_{\Gamma_0}| \leq \frac{\epsilon}{3}, \tag{3}$$

$$|J^*_{C_0} - J^*_{\Gamma_0}| \leq \frac{\epsilon}{3}. \tag{4}$$

Then relations (1), (2), (3), and (4) allow us to write

$$|J^*_C - J_{C_0}| \leq |J^*_C - J^*_{C_0}| + |J^*_{C_0} - J^*_{\Gamma_0}| + |J_{\Gamma_0} - J_{C_0}| \leq \epsilon \tag{5}$$

if the curve C lies in the ρ-neighborhood of the curve C_0. In particular it follows from relation (5) that

$$J^*_C \geq J_{C_0} - \epsilon. \tag{6}$$

Now consider the difference

$$J_C - J^*_C$$

$$= \sum_{i=1}^{m} \int_{C^i} \{F(x, y, x', y') - x'F_p(x, y, \alpha_i, \beta_i) - y'F_q(x, y, \alpha_i, \beta_i)\}\, ds$$

$$= \sum_{i=1}^{m} \int_{C^i} E(x, y, \alpha_i, \beta_i, x', y')\, ds.$$

The functional J_C is quasi-regular, thus $E \geq 0$. Hence

$$J_C \geq J^*_C$$

and if we take relation (6) into account we have

$$J_C \geq J_{C_0} - \epsilon$$

that is, we have established lower semicontinuity.

The existence of a broken line that satisfies inequalities (3) and (4) remains to be demonstrated. We inscribe in the curve C_0 a sequence of broken lines $\{\Gamma_n\}_1^\infty$ that have C_0 as limit curve. Suppose λ_n is the perimeter of Γ_n and s_n is the arc length on Γ_n measured from the initial point on the line Γ_n to a point on this line. We suppose that

$$x = f_n(s_n), \qquad y = g_n(s_n)$$

is the equation of Γ_n and

$$x = \phi_0(s), \qquad y = \psi_0(s), \qquad (0 \leq s \leq l)$$

is the equation of the curve C_0. Setting

$$s_n = \frac{\lambda_n}{l} s$$

we obtain the equation of the broken line in the form

$$x = f_n\left(\frac{\lambda_n s}{l}\right) \equiv \phi_n(s), \qquad y = g_n\left(\frac{\lambda_n s}{l}\right) \equiv \psi_n(s), \qquad (0 \leq s \leq l).$$

Then the following equations hold

$$\int_0^l \{[\phi_0'(s)]^2 + [\psi_0'(s)]^2\}\, ds = l, \tag{7}$$

$$\int_0^l \{[\phi_n'(s)]^2 + [\psi_n'(s)]^2\}\, ds = \frac{\lambda_n}{l} \int_0^{\lambda_n} \{[f_n'(s_n)]^2 + [g_n'(s_n)]^2\}\, ds_n = \frac{\lambda_n^2}{l}. \tag{8}$$

We note here that each of the sequences $\{\phi_n(s)\}_1^\infty$, $\{\psi_n(s)\}_1^\infty$ satisfies the conditions of Arzela's theorem. Hence from the pair of sequences $\{\phi_n(s), \psi_n(s)\}_1^\infty$ we can choose a pair of subsequences (we shall denote these also by $\{\phi_n(s), \psi_n(s)\}_1^\infty$) which converge uniformly to a pair of functions and this pair is obviously $\{\phi_0(s), \psi_0(s)\}$. Besides this, relations (7) and (8) provide that all

the integrals

$$\int_0^l \{[\phi_n'(s)]^2 + [\psi_n'(s)]^2\}\, ds,$$

$$\int_0^l \{[\phi_0'(s)]^2 + [\psi_0'(s)]^2\}\, ds, \qquad (n = 1, 2, \cdots)$$

do not exceed one and the same finite value. These two properties of the pair of sequences $\{\phi_n(s), \psi_n(s)\}_1^\infty$ enable us to establish, with the aid of Section 29, the fact that the pair of sequences $\{\phi_n'(s), \psi_n'(s)\}_1^\infty$ converges weakly to the pair $\{\phi_0'(s), \psi_0'(s)\}$. Here we use the term weak convergence to mean that for any two measurable functions $\xi(s)$, $\eta(s)$ for which the relation

$$\int_0^l \{[\xi(s)]^2 + [\eta(s)]^2\}\, ds < \infty$$

holds, the relation

$$\lim_{n\to\infty} \int_0^l \{\phi_n'(s)\xi(s) + \psi_n'(s)\eta(s)\}\, ds = \int_0^l \{\phi_0'(s)\xi(s) + \psi_0'(s)\eta(s)\}\, ds$$

also holds.

Having demonstrated that the pair of sequences $\{\phi_n'(s), \psi_n'(s)\}_1^\infty$ converges weakly to the pair $\{\phi_0'(s), \psi_0'(s)\}$, we can use relations (7) and (8) as well as the relation

$$\lim_{n\to\infty} \lambda_n = l \tag{9}$$

to show that

$$\lim_{n\to\infty} \int_0^l \{[\phi_n'(s) - \phi_0'(s)]^2 + [\psi_n'(s) - \psi_0'(s)]^2\}\, ds = 0.$$

This equation shows that the pair of sequences $\{\phi_n'(s), \psi_n'(s)\}_1^\infty$ *converges strongly* to the pair $\{\phi_0'(s), \psi_0'(s)\}$.

In the right side of the equation

$$\int_0^l \{[\phi_n'(s) - \phi_0'(s)]^2 + [\psi_n'(s) - \psi_0'(s)]^2\}\, ds$$

$$= \frac{\lambda_n^2}{l} + l - 2\int_0^l \{\phi_n'(s)\phi_0'(s) + \psi_n'(s)\psi_0'(s)\}\, ds$$

equation (9) implies that the first term converges to l and the third term converges to

$$-2\int_0^l \{[\phi_0'(s)]^2 + [\psi_0'(s)]^2\}\, ds = -2l$$

because the pair $\{\phi_n'(s), \psi_n'(s)\}_1^\infty$ converges weakly to the pair $\{\phi_0'(s), \psi_0'(s)\}$. Hence the right member converges to zero; this shows that the pair $\{\phi_n'(s), \psi_n'(s)\}_1^\infty$ converges strongly to $\{\phi_0'(s), \psi_0'(s)\}$.

It is now easy to show that

$$\lim_{n\to\infty} J_{\Gamma_n} = J_{C_0}. \tag{10}$$

We write

$$\begin{aligned} J_{\Gamma_n} - J_{C_0} &= \int_0^l F(\phi_n, \psi_n, \phi_n', \psi_n')\, ds - \int_0^l F(\phi_0, \psi_0, \phi_0', \psi_0')\, ds \\ &= \int_0^l \{F(\phi_n, \psi_n, \phi_n', \psi_n') - F(\phi_0, \psi_0, \phi_n', \psi_n')\}\, ds \\ &+ \int_0^l \{F(\phi_0, \psi_0, \phi_n', \psi_n') - F(\phi_0, \psi_0, \phi_0', \psi_0')\}\, ds \\ &= K_1 + K_2. \end{aligned}$$

As $n \to \infty$, the first term K_1 has the limit zero because $\{\phi_n(s), \psi_n(s)\}_1^\infty$ converges uniformly to $\{\phi_0(s), \psi_0(s)\}$. The second term K_2 can be put in the form

$$K_2 = \int_0^l \{(\phi_n' - \phi_0')\tilde{F}_p + (\psi_n' - \psi_0')\tilde{F}_q\}\, ds$$

in which

$$\begin{aligned} \tilde{F}_p &= F_p(\phi_0, \psi_0, \phi_0' + \theta(\phi_n' - \phi_0'), \psi_0' + \theta(\psi_n' - \psi_0')), \\ \tilde{F}_q &= F_q(\phi_0, \psi_0, \phi_0' + \theta(\phi_n' - \phi_0'), \psi_0' + \theta(\psi_n' - \psi_0')). \end{aligned}$$

It follows that

$$|K_2| \leq A \sqrt{\int_0^l \{[\phi_n' - \phi_0']^2 + [\psi_n' - \psi_0']^2\}\, ds}$$

in which A is a constant. Hence K_2 approaches zero because the pair of sequences $\{\phi_n'(s), \psi_n'(s)\}_1^\infty$ converges strongly to the pair $\{\phi_0'(s), \psi_0'(s)\}$. Thus relation (10) has been proved. It follows that

for $n > N_1(\epsilon)$

$$|J_{C_0} - J_{\Gamma_n}| \leq \frac{\epsilon}{3}.$$

Now we must demonstrate that for $n > N_2(\epsilon)$

$$|J^*_{C_0} - J^*_{\Gamma_n}| \leq \frac{\epsilon}{3}. \tag{11}$$

To show this we take a positive number $\delta \leq \epsilon/12l$. We can find a $\rho = \rho(\delta)$ such that in every circle of radius ρ the oscillation of neither the function $F_p(x, y, \cos\theta, \sin\theta) = P_\theta(x, y)$ nor the function $F_q(x, y, \cos\theta, \sin\theta) = Q_\theta(x, y)$ will exceed δ for arbitrary θ. Now take $N_3(\rho) = N_2(\epsilon)$ sufficiently large that, for $n > N_3(\rho)$, each segment of the broken line Γ_n with the subtended arc of the curve C_0 lies in a circle of radius ρ with its center on the segment considered. We designate the coordinates of the center for the jth segment by x_j, y_j and the slope of the jth segment by θ_j; then we write

$$p_j = P_{\theta_j}(x_j, y_j), \qquad q_j = Q_{\theta_j}(x_j, y_j).$$

In so far as

$$\oint \{p_j x' + q_j y'\}\, ds = 0$$

we can write

$$\int_{C_0^j} \{p_j x' + q_j y'\}\, ds = \int_{\Gamma_n^j} \{p_j x' + q_j y'\}\, ds.$$

On the other hand

$$\left| J^*_{C_0} - \sum_j \int_{C_0^j} \{p_j x' + q_j y'\}\, ds \right| \leq 2l\delta,$$

$$\left| J^*_{\Gamma_n} - \sum_j \int_{\Gamma_n^j} \{p_j x' + q_j y'\}\, ds \right| \leq 2l\delta$$

which implies that

$$|J^*_{C_0} - J^*_{\Gamma_n}| \leq 4l\delta \leq \frac{\epsilon}{3}.$$

This completes the proof of inequality (11) and that of Tonelli's theorem, Section 33.

EXERCISES

1. Using the method of Ritz, find an approximate solution for the problem of finding a minimum for the functional

$$\int_0^1 (y'^2 - y^2 - 2xy)\, dx; \qquad y(0) = y(1) = 0.$$

Compare this approximation with the exact solution:

$$y = \frac{\sin x}{\sin 1} - x.$$

2. Find an approximate solution for the problem of finding an extremum for

$$\int_0^1 (x^3 y''^2 + 100xy^2 - 20xy)\, dx; \qquad y(1) = y'(1) = 0.$$

Appendix

Supplementary Topics and Exercises

A–1. (*See Sections* 4–7 *and* 19.) We wish to find an extremal for the functional

$$\int \sqrt{x^2 + y^2}\,\sqrt{1 + y'^2}\,dx.$$

SOLUTION:

A. The functional is regular in any part of the plane that does not contain the origin. Hilbert's theorem applies here and this permits us to restrict ourselves to a consideration of the Euler equation in differentiated form.

It is advantageous to change from Cartesian coordinates to polar coordinates and to make use of the invariance of the Euler equation. In polar coordinates the functional becomes

$$\int r\sqrt{r^2 + r'^2}\,d\phi.$$

Immediately we have a first integral of the Euler equation:

$$\frac{r^3}{\sqrt{r^2 + r'^2}} = \alpha.$$

Further integration gives

$$r^2 \cos(2\phi - \beta) = \alpha$$

and after transforming to the original coordinates

$$x^2 \cos\beta + 2xy \sin\beta - y^2 \cos\beta = \alpha \tag{1}$$

in which α and β are arbitrary constants.

B. Let us use the canonical variables. If we set

$$v = \sqrt{x^2 + y^2}\,\frac{y'}{\sqrt{1 + y'^2}},$$

$$\begin{aligned} H(x, y, v) &= \sqrt{x^2 + y^2}\,\sqrt{1 + y'^2} - vy' \\ &= \sqrt{x^2 + y^2 - v^2} \end{aligned}$$

we obtain the canonical equations

$$\frac{dv}{dx} = \frac{y}{\sqrt{x^2 + y^2 - v^2}}, \qquad \frac{dy}{dx} = \frac{v}{\sqrt{x^2 + y^2 - v^2}}.$$

A first integral of this system has the form

$$v^2 - y^2 = A^2.$$

Hence

$$\frac{dy}{dx} = \frac{\sqrt{y^2 + A^2}}{\sqrt{x^2 - A^2}}$$

and then

$$y + \sqrt{y^2 + A^2} = B(x + \sqrt{x^2 - A^2})$$

which implies that

$$x^2 + \left(B - \frac{1}{B}\right)xy - y^2 = \frac{1}{4}\left(B + \frac{1}{B}\right)^2 A^2$$

and if we set

$$\cos\beta = \frac{2B}{B^2 + 1}, \qquad \sin\beta = \frac{B^2 - 1}{B^2 + 1}, \qquad \alpha = \frac{(B^2 + 1)A^2}{2B}$$

we arrive at the solution (1).

C. We can make use of the Hamilton-Jacobi equation

$$\frac{\partial W}{\partial x} = \sqrt{x^2 + y^2 - \left(\frac{\partial W}{\partial y}\right)^2}$$

or

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = x^2 + y^2.$$

It is natural to write the solution in the form $W = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2)$. Substitution and comparison of coefficients yields

$$A^2 + B^2 = 1, \qquad B(A + C) = 0, \qquad B^2 + C^2 = 1.$$

Thus we can set

$$A = -C = \sin \beta, \qquad B = -\cos \beta.$$

The function

$$W = \tfrac{1}{2}(x^2 \sin \beta - 2xy \cos \beta - y^2 \sin \beta)$$

is a solution of the Hamilton-Jacobi equation satisfying the conditions of Theorem 2, Section 19. Hence a general integral of the Euler equation has the form

$$\frac{\partial W}{\partial \beta} = \frac{1}{2}\alpha.$$

This coincides with the solution (1).

A–2. (*See Sections* 4 *and* 9.) Show that through any two points of the plane with different abscissas there is one and only one extremal for each of the functionals

$$\int e^{-2y^2}(y'^2 - 1)\,dx, \qquad \int \{y^2 + y' \arctan y' - \ln \sqrt{1 + y'^2}\}\,dx.$$

SOLUTION: These are examples for the application of the theorem of S. N. Bernstein, Section 9, since the Euler equation for each functional has the form

$$y'' = 2y(1 + y'^2).$$

A–3. (*See Sections* 4 *and* 9.) Show that it may be impossible to draw an extremal through any two points with different abscissas for the functional*

$$\int \{y^2 + \sqrt{1 + y'^2}\}\,dx.$$

SOLUTION: The Euler equation has the form

$$y'' = 2y(1 + y'^2)^{3/2}$$

*This example is due to S. N. Bernstein; see the reference in Section 9.

and S. N. Bernstein's theorem does not apply; this, however, is not why an extremal cannot be drawn through any two points with different abscissas.

A first integral of the Euler equation is

$$(C - y^2)\sqrt{1 + y'^2} = 1$$

from which we find that

$$\frac{dy}{dx} = \frac{\sqrt{1 - (y^2 - C)^2}}{y^2 - C}$$

in which C is a real constant. For such values of the constant C, this expression will not be real for all values of y in the interval $[a_1, b_1]$ if, for example $a_1 = 0$, and $b_1 > \sqrt{2}$.

A–4. (*See Sections* 4 *and* 18.) For the functional

$$\int y^m g(y')\, dx$$

the general integral of the Euler equation has the form

$$y = \alpha\phi\left(\frac{x - \beta}{\alpha}\right)$$

in which α and β are arbitrary constants; thus all extremals can be obtained from one,

$$y = \phi(x)$$

by means of a similarity transformation with center at the origin followed by a translation parallel to the x-axis.

If M and M^* are conjugate points on an extremal then tangents to the extremal at these points intersect on the x-axis.

A–5. (*See Sections* 4 *and* 27.) Determine whether it is possible to show that every equation

$$y'' = F(x, y, y')$$

is the Euler equation for some functional

$$\int f(x, y, y')\, dx.$$

How do we determine the function $f(x, y, y')$ from the function $F(x, y, y')$?

Determine whether it is possible to find all functionals for which the extremals are

A. all straight lines $y = C_1x + C_2$

B. all parabolas $y = C_1x + C_2x^2$.

SOLUTION: We are seeking the functional

$$\int f(x, y, y')\, dx$$

for which the Euler equation

$$f_y - \frac{d}{dx} f_{y'} = 0$$

or

$$f_y - f_{y'x} - f_{y'y}y' - f_{y'y'}y'' = 0$$

coincides with the equation

$$y'' = F(x, y, y').$$

This implies that we must have an identity in x, y, y'

$$f_y - f_{y'x} - f_{y'y}y' - f_{y'y'}F = 0.$$

Differentiation with respect to y' yields

$$f_{y'y'x} + f_{y'y'y}y' + f_{y'y'y'}F + f_{y'y'}F_{y'} = 0.$$

From this we find a partial differential equation for the function

$$u = f_{y'y'}.$$

It is

$$\frac{\partial u}{\partial x} + y' \frac{\partial u}{\partial y} + F \frac{\partial u}{\partial y'} + F_{y'}u = 0.$$

Thus finding the functional leads to integration of a linear partial differential equation and a quadrature.

In case A, the Euler equation is $y'' = 0$ and for u we have the equation

$$\frac{\partial u}{\partial x} + y' \frac{\partial u}{\partial y} = 0.$$

Integration yields

$$u = \Phi(y - xy', y')$$

in which Φ is an arbitrary function of its arguments. From this it follows that

$$f(x, y, z) = \alpha(x, y) + z\beta(x, y) + \int_0^z (z - t)\Phi(y - tx, t)\, dt.$$

Here $\alpha(x, y)$ and $\beta(x, y)$ are arbitrary functions of their arguments and satisfy the relation

$$\frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial x} = 0.$$

In case B the Euler equation has the form

$$y'' = \frac{2(xy' - y)}{x^2}.$$

The equation for u is

$$\frac{\partial u}{\partial x} + y' \frac{\partial u}{\partial y} + \frac{2(xy' - y)}{x^2} \frac{\partial u}{\partial y'} + \frac{2}{x} u = 0.$$

We are seeking a solution of the form

$$v(x, y, y', u) = 0.$$

As an equation for v we have

$$\frac{\partial v}{\partial x} + y' \frac{\partial v}{\partial y} + \frac{2(xy' - y)}{x^2} \frac{\partial v}{\partial y'} - \frac{2u}{x} \frac{\partial v}{\partial u} = 0.$$

We solve the system

$$\frac{dx}{1} = \frac{dy}{y'} = \frac{dy'}{2(xy' - y)/x^2} = \frac{du}{-2u/x}.$$

Then the first integrals have the form

$$ux^2 = C_1, \qquad \frac{xy' - y}{x^2} = C_2, \qquad -y' + \frac{2y}{x} = C_3.$$

From this we obtain the following expression for v:

$$v = \Psi\left(ux^2, \frac{xy' - y}{x^2}, \frac{2y}{x} - y'\right)$$

in which Ψ is an arbitrary function. Hence

$$u = \frac{1}{x^2}\Phi\left(\frac{xy' - y}{x^2}, \frac{2y}{x} - y'\right)$$

in which Φ is an arbitrary function of its arguments. From this

$$f(x, y, z) = \alpha(x, y) + z\beta(x, y) + \frac{1}{x^2}\int_{y/x}^{z}(z - t)\Phi\left(\frac{xt - y}{x^2}, \frac{2y}{x} - t\right)dt$$

in which $\alpha(x, y)$ and $\beta(x, y)$ are arbitrary functions of their arguments and satisfy the relation

$$\frac{\partial\alpha}{\partial y} - \frac{\partial\beta}{\partial x} = 0.$$

A–6. (*See Section* 19.) Show that a functional can be found for the Hamilton-Jacobi equation

$$\frac{\partial W}{\partial x} = H\left(x, y, \frac{\partial W}{\partial y}\right).$$

Our problem is to define the functional if the Hamilton-Jacobi equation has either of the forms

A. $$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = 1$$

B. $$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = x^2 + y^2.$$

SOLUTION: We begin by considering the relations

$$H(x, y, v) = f - y'f_{y'} \qquad v = f_{y'}.$$

We derive a Clairaut equation from these to define $f(x, y, z)$ as a

function of z:

$$f - zf_z = H(x, y, f_z)$$

(x and y act as parameters).

In case A, this equation has the form

$$f - zf_z = \sqrt{1 - f_z^2}$$

which yields

$$f = \sqrt{1 + z^2}.$$

In case B, we obtain the equation

$$f - zf_z = \sqrt{x^2 + y^2 - f_z^2}$$

which yields

$$f = \sqrt{x^2 + y^2}\,\sqrt{1 + z^2}.$$

A–7. (*See Sections* 4, 18, *and* 27.) The problem* here is to show that the greatest lower bound of the values of the functional

$$J[y] = \int_{-1}^{1} x^2 y'^2\, dx, \qquad [y(-1) = -1, y(1) = 1]$$

is zero. Hence this functional does not have a minimum in the class of piecewise-smooth functions.

SOLUTION: For the proof we take the admissible function

$$y_a(x) = \frac{\arctan\,(x/a)}{\arctan\,(1/a)}, \qquad (a > 0).$$

Then

$$J[y_a] = \frac{a^2}{[\arctan\,(1/a)]^2} \int_{-1}^{1} \frac{x^2 dx}{(x^2 + a^2)^2}$$

$$< \frac{a^2}{[\arctan\,(1/a)]^2} \int_{-1}^{1} \frac{dx}{x^2 + a^2} = \frac{2a}{\arctan\,1/a}.$$

As $a \to 0$, the right side approaches zero.

---

*This example is due to Weierstrass.

A–8. (*See Sections* 4 *and* 18.) Find a minimum for the functional

$$J[y] = \int_{-1}^{1} x^{\frac{2}{3}} y'^2 \, dx \qquad [y(-1) = -1, y(1) = 1].$$

SOLUTION: The Euler equation has a first integral

$$x^{\frac{2}{3}} y' = \tfrac{1}{3} C.$$

Integration yields

$$y = Cx^{\frac{1}{3}} + D.$$

The curve

$$y = \bar{y}(x) \equiv x^{\frac{1}{3}}$$

passes through the points $(-1, -1)$, $(1, 1)$.

The general theory does not apply because $\bar{y}(x)$ is not a piecewise-smooth function. However, suppose that y is an arbitrary function which satisfies the conditions at the end points, has a derivative almost everywhere, and is such that the integral exists. Then, by comparing $J[\bar{y}]$ and $J[y]$, we see immediately that $\bar{y}(x)$ furnishes an absolute minimum.

A–9. (*See Section* 15.) For the functional

$$\int_a^b n(x, y)\sqrt{1 + y'^2} \, dx$$

where $n(x, y) \geq 0$ everywhere in G the Weierstrass E function $E(x, y, z, q)$ is nonnegative for arbitrary finite z and q at any point of G. Consequently there is no question as to whether the Weierstrass condition is satisfied.

A–10. (*See Section* 18.) For the functional

$$\int_a^b f(x, y') \, dx$$

where f does not contain y explicitly, every extremal can always be imbedded in a field so that there is no need to check the Jacobi condition.

A–11. (*See Sections* 15, 17, *and* 18.) Find extrema for the functional

$$J[y] = \int_a^b y'^2(1 + y')^2\,dx, \qquad [y(a) = a_1, y(b) = b_1].$$

SOLUTION: The extremals are straight lines. Suppose

$$y = \bar{y}(x) \equiv mx + n$$

is the equation of an extremal joining the points (a, a_1) and (b, b_1). Then

$$J[\bar{y}] = m^2(1 + m)^2(b - a).$$

The family of extremals

$$y = mx + \beta$$

represents a field; its slope is m.

The Weierstrass function has the form

$$E(x, y, m, q) = (q - m)^2\{q^2 + 2(1 + m)q + (1 + m)(1 + 3m)\}.$$

The discriminant of the quantity in braces is

$$D = -2m(1 + m).$$

For $-1 < m < 0$ it is positive. Consequently for m in this interval the Weierstrass E function assumes both positive and negative values. Hence there is no strong extremum for $-1 < m < 0$; however for $m \geq 0$ and $m \leq -1$ we have a strong minimum.

To verify the Legendre condition consider the function

$$f_{y'y'}(x, y, z) = 2(1 + 6z + 6z^2).$$

The zeros of this quadratic are equal to

$$m_1 = \tfrac{1}{2}(-1 + 1/\sqrt{3}) \qquad \text{and} \qquad m_2 = \tfrac{1}{2}(-1 - 1/\sqrt{3}).$$

If $m_1 < m < 0$ or $-1 < m < m_2$, then

$$f_{y'y'}(x, \bar{y}, \bar{y}') > 0$$

which means that we have a weak minimum. For $m_2 < m < m_1$ we have a weak maximum. Finally for $m = m_1$ or $m = m_2$ the theory does not apply but an immediate verification shows that

there are no extrema. Indeed, suppose that

$$y = \bar{y} + \omega$$

then

$$J[y] - J[\bar{y}] = 6(m - m_1)(m - m_2)\int_a^b \omega'^2\,dx + 2(2m + 1)\int_a^b \omega'^3\,dx + \int_a^b \omega'^4\,dx.$$

Consequently, for $m = m_1$ or $m = m_2$

$$J[y] - J[\bar{y}] = 2(2m + 1)\int_a^b \omega'^3\,dx + \int_a^b \omega'^4\,dx$$

that is, for sufficiently small $|\omega'|$ the sign of the quantity

$$\Delta J = J[y] - J[\bar{y}]$$

can be positive or negative.

A–12. (*See Sections* 15, 17, *and* 18.) Determine whether or not the extremal

$$y = \bar{y}(x) \equiv 0$$

furnishes a minimum for the functional $(a > 0, b > 0)$

$$J[y] = \int_0^1 (ay'^2 - 4byy'^3 + 2bxy'^4)\,dx, \qquad [y(0) = 0, y(1) = 0].$$

SOLUTION: The family of straight lines $y = \text{const}$ represents a field in which $y = 0$ is imbedded. The Legendre condition is satisfied in strengthened form, namely, for arbitrary finite q

$$f_{y'y'}(x, \bar{y}(x), q) = 2a + 24bxq^2 \geq 2a > 0.$$

Hence the necessary Weierstrass condition is satisfied. This follows at once:

$$E(x, \bar{y}(x), \bar{y}'(x), q) = q^2(a + 2bxq^2) \geq 0.$$

However, there is no strong minimum; if we take the broken line

$$y_0(x) = \begin{cases} \dfrac{k}{h}x, & (0 \leq x \leq h) \\ k\dfrac{1 - x}{1 - h}, & (h \leq x \leq 1) \end{cases}$$

then for $k > 0$ sufficiently small $h > 0$ can be chosen so that $J[y_0] < 0$.

A–13. (*See Sections* 15, 17, *and* 18.) Find a minimum for the functional

$$J[y] = \int_1^2 (xy'^4 - 2yy'^3)\,dx, \qquad [y(1) = 0, y(2) = 1]$$

and investigate its nature.

SOLUTION: The extremals are straight lines. The extremal $y = \bar{y}(x) \equiv x - 1$ satisfies the given boundary conditions.

The family of extremals

$$y = x + \beta$$

furnishes a field; its slope is 1. The Weierstrass E function is nonnegative;

$$E(x, y, 1, q) = (q - 1)^2 \{xq^2 + 2(x - y)q + (3x - 4y)\} \geq 0$$

for $x > 0$, $-1 - \sqrt{3} < y/x < -1 + \sqrt{3}$. Hence there is a strong minimum which can be verified by calculating the difference of the functionals

$$\Delta J = J[\bar{y} + \omega] - J[\bar{y}]$$
$$= \int_1^2 \omega'^2 \{x\omega'^2 + 2(1 + x - \omega)\omega' + 6(1 - \omega)\}\,dx.$$

The integrand is nonnegative if $|\omega| \leq \frac{1}{3}$, $(1 \leq x \leq 2)$.

A–14. (*See Sections* 15 *and* 17.) In order that an extremal $y = \bar{y}(x)$ imbedded in a field furnish a strong minimum it is sufficient that

$$f_{y'y'}(x, y, q) > 0 \tag{1}$$

for arbitrary finite q and for (x, y) in some neighborhood of the extremal $y = \bar{y}(x)$. Is this condition necessary?

In order that an extremal imbedded in a field furnish a strong minimum it is necessary that

$$E(x, \bar{y}(x), \bar{y}'(x), q) \geq 0 \tag{2}$$

for arbitrary finite q. Is this condition sufficient?

SOLUTION: Section 11 of the Appendix shows that inequality (1) is not necessary for a strong minimum. In this example

$$f_{y'y'}(x, y, z) = 2(1 + 6z + 6z^2)$$

does not preserve its sign for all finite z; besides this, for certain slopes, the extremal will furnish a strong minimum. Section 12 of the Appendix shows that inequality (2) is not sufficient for a strong minimum; we showed that there was no strong minimum although

$$E(x, \bar{y}(x), \bar{y}'(x), q) = q^2(a + 2bxq^2) \geq 0.$$

A–15. MINIMUM SURFACE OF REVOLUTION. (*See Sections* 10, 15, *and* 18.) Suppose we have two points $A(a, a_1)$, $B(b, b_1)$ above the x-axis. The problem is to join these two points by a curved line so that when this curve is revolved about the x-axis a surface of least area is generated.

SOLUTION: We restrict the solution to the case in which both points are at the same altitude and take the y-axis so that it is equidistant from both points. We may assume that the coordinates of the points A, B are $(-1, h)$ and $(1, h)$ respectively. An admissible curve lies in the half plane $y \geq 0$.

We take the functional to be minimized in parametric form:

$$J_C = 2\pi \int_{t_1}^{t_2} y\sqrt{\dot{y}^2 + \dot{x}^2}\, dt.$$

The first Euler equation has as its first integral

$$\frac{y\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C \tag{1}$$

and the second Euler equation has the form

$$\frac{d}{dt}\frac{y\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - \sqrt{\dot{x}^2 + \dot{y}^2} = 0.$$

This shows that the straight lines

$$y = \text{const}$$

are not extremals. The straight lines

$$x = \text{const} \tag{2}$$

are evidently extremals.

To obtain further extremals we write equation (1) in the form

$$y\frac{dx}{ds} = C$$

in which s is arc length and integrate this equation. Integral curves distinct from the lines $x = \text{const}$ are catenaries

$$y = C\cosh\frac{x - D}{C}. \tag{3}$$

First we must ascertain whether or not there is a curve of the family (3) that passes through points A and B. To answer this question we must solve the equation

$$C\cosh\frac{1}{C} = h$$

for C. We construct the graph of the function

$$\phi(C) = C\cosh\frac{1}{C}$$

for $C > 0$. The graph is concave upward since

$$\phi'(C) = \cosh\frac{1}{C} - \frac{1}{C}\sinh\frac{1}{C}$$

and consequently

$$\phi''(C) = \frac{1}{C^3}\cosh\frac{1}{C} > 0.$$

Suppose the curve has a minimum at the point C_0 as in Figure 6; then C_0 is a root of the equation

$$\xi = \tanh\frac{1}{\xi}.$$

The approximate value of C_0 is

$$C_0 = 0.83355.$$

Consequently

$$\phi(C_0) = \min \phi(C) = \frac{C_0}{\sqrt{1 - C_0^2}} = 1.5127.$$

If $h < \phi(C_0)$, an extremal that joins the points A and B does not exist. If $h = \phi(C_0)$ there will be one extremal and if $h > \phi(C_0)$

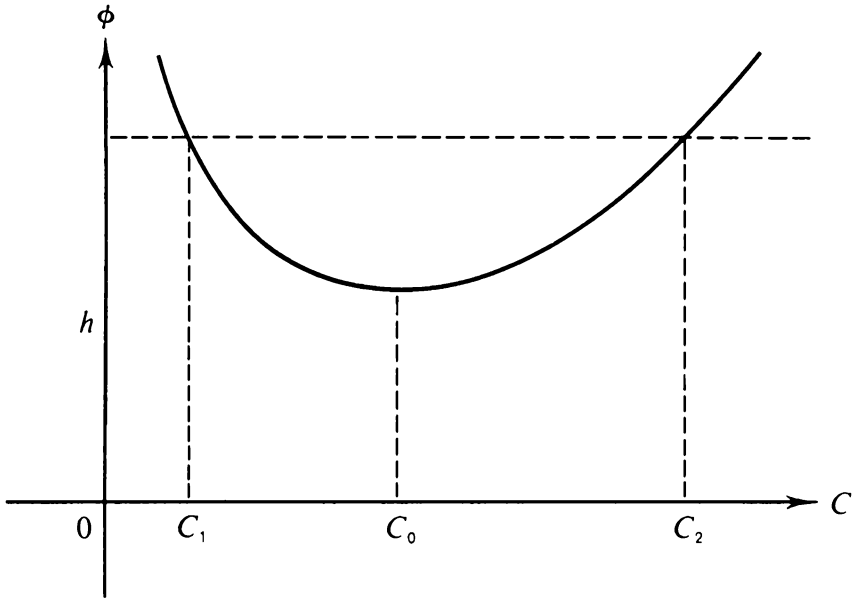

Figure 6

there will be two such extremals. If $h > \phi(C_0)$ we will get two values for C: to the smaller value (C_1) will correspond a lower catenary and to the larger value (C_2) will correspond a higher catenary. We shall suppose that $h > \phi(C_0)$ and determine whether the Jacobi condition is satisfied; it is not necessary to check the Weierstrass condition here (see Section 9 of the Appendix).

The linearly independent solutions of the Jacobi equation have the form

$$u_1 = \sinh \frac{x}{C}, \qquad u_2 = \cosh \frac{x}{C} - \frac{x}{C} \sinh \frac{x}{C}.$$

Hence

$$\Delta(x; -1) = \left(\cosh\frac{x}{C} - \frac{x}{C}\sinh\frac{x}{C}\right)\sinh\frac{1}{C} + \left(\cosh\frac{1}{C} - \frac{1}{C}\sinh\frac{1}{C}\right)\sinh\frac{x}{C}.$$

The point $x = 0$ is not a zero of this function. Hence, rather than

$$\Delta(x; -1),$$

it is sufficient to consider the function

$$z(x) = \frac{\Delta(x; -1)}{\sinh(1/C)\cdot\sinh(x/C)} = \left(\coth\frac{x}{C} - \frac{x}{C}\right) + \left(\coth\frac{1}{C} - \frac{1}{C}\right).$$

This function decreases in each of the half-open intervals $[-1, 0)$ and $(0, 1]$ because

$$z'(x) = -\frac{1}{C}\frac{1}{\sinh^2(x/C)} - \frac{1}{C} < 0.$$

Furthermore,

$$z(1) = \frac{2}{\sinh(1/C)}\left(\cosh\frac{1}{C} - \frac{1}{C}\sinh\frac{1}{C}\right) = \frac{2}{\sinh(1/C)}\phi'(C)$$

so that

$$z(1) < 0 \text{ if } C = C_1$$

and

$$z(1) > 0 \text{ if } C = C_2.$$

Hence the graph of the function $z(x)$ appears as in Figure 7.

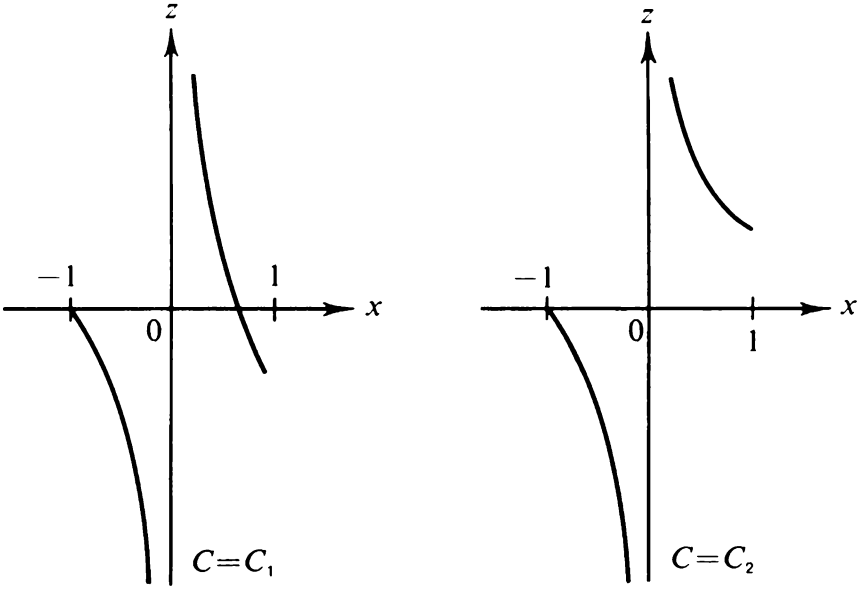

Figure 7

We see that the Jacobi condition is not satisfied for the lower catenary ($C = C_1$) but for the upper one ($C = C_2$) it is satisfied. Thus the lower catenary will not furnish a minimum, but the upper one will furnish a strong minimum.

Now we consider the discontinuous solution to which we must have recourse if

$$h < \phi(C_0).$$

In this case, the given points can not be joined by an extremal. We put aside the condition that the points A and B are at the same height and suppose that their coordinates are (a, a_1), (b, b_1). A broken line consisting of an arc of a catenary (3) and a straight line (2) will not satisfy the Weierstrass-Erdmann corner condition. Hence we shall consider a broken line composed of extremals and part of the boundary of the region, that is, the x-axis. It is evident that we should take segments of the lines $x = a, x = b$ as extremals. Upon being rotated about the x-axis, the broken line will give two circles joined by the x-axis. We can prove immediately that this broken line will furnish a strong minimum. Every curve L joining the points A and B and lying in a strong neighborhood of

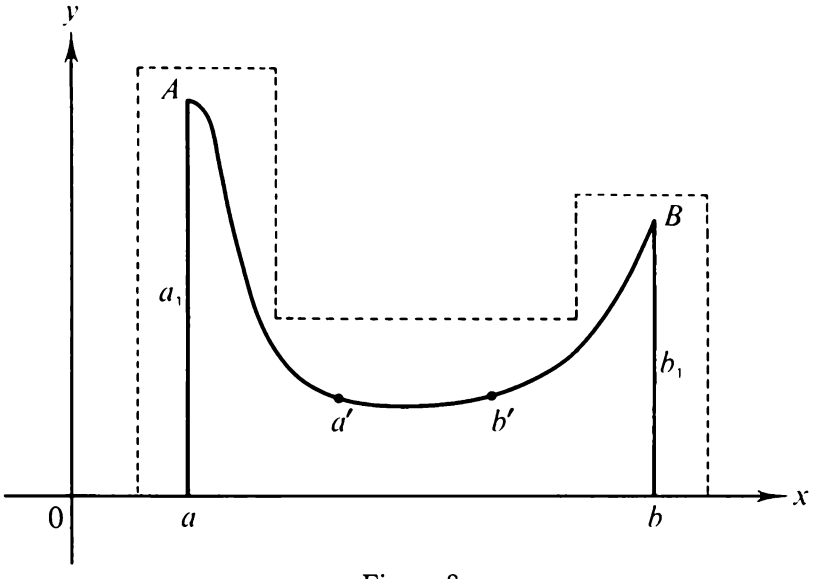

Figure 8

the broken line $AabB$ as shown in Figure 8 has length that exceeds $Aa + bB = a_1 + b_1$. Suppose that the points a' and b' on L are situated so that $Aa' = Aa$ and $b'B = bB$. Then of two points on aA and $a'A$ equidistant from A as measured along aA and $a'A$ respectively the second will be higher. Hence

$$\int_{aA} y\,ds < \int_{a'A} y\,ds$$

and analogously

$$\int_{bB} y\,ds < \int_{b'B} y\,ds.$$

Consequently

$$\int_{L} y\,ds > \int_{AabB} y\,ds.$$

This relation shows that for $a_1 + b_1 < m$ in which m is the distance between the points A and B the broken line $AabB$ will give an absolute minimum.

A–16. (*See Sections* 4, 15, *and* 18.) Find a minimum for the functional

$$J[y] = \int_0^m \sqrt{y + h}\,\sqrt{1 + y'^2}\,dx$$

in which $h > 0$ and $y(0) = 0, y(m) = m_1 > -h$.

SOLUTION: A first integral of the Euler equation has the form

$$\frac{\sqrt{y + h}}{\sqrt{1 + y'^2}} = C.$$

Further integration yields $x - A = 2C\sqrt{y + h - C^2}$ or $(x - A)^2 = 4C^2(y + h - C^2)$. Let us take a pencil of extremals through the point $(0, 0)$. For this

$$A^2 = 4C^2(h - C^2) \tag{1}$$

and consequently

$$x^2 - 2Ax = 4C^2y. \tag{2}$$

We introduce as parameter the slope of the extremal at the origin

$$y'(0) = \alpha.$$

Then

$$4\alpha C^2 = -2A. \tag{3}$$

Systems (1) and (3) yield

$$C^2 = \frac{h}{1 + \alpha^2} \qquad \text{and} \qquad A = -\frac{2\alpha h}{1 + \alpha^2}.$$

The equation of the pencil (2) will have the form

$$y = \alpha x + \frac{1 + \alpha^2}{4h}x^2. \tag{4}$$

To find the envelope of this family of parabolas, we differentiate with respect to α; this yields

$$0 = x + \frac{\alpha x^2}{2h}.$$

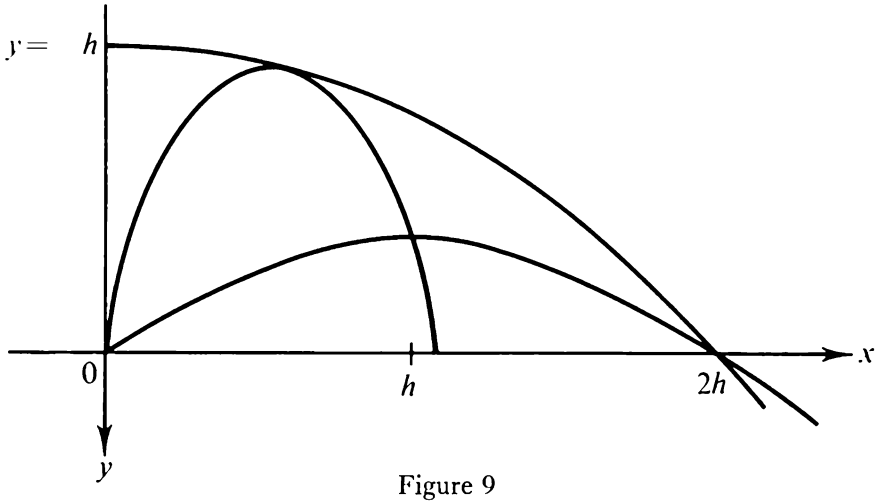

Figure 9

The equation of the envelope is

$$y = -h + \frac{x^2}{4h}. \tag{5}$$

All the extremals (4) lie under the parabola (5) which in ballistics is called the parabola of safety, Figure 9. If the point (m, m_1) lies outside the parabola (5) then it cannot be joined to the origin by an extremal. If the point (m, m_1) lies below the parabola of safety then we can find two parabolas of the pencil (4) that join (m, m_1) to the origin. The Weierstrass condition does not need to be tested and the Jacobi condition is easily checked with the aid of a geometric interpretation of conjugate points. Conclusion: the upper parabola does not furnish a minimum but the lower one furnishes a strong minimum.

A–17. THE BRACHISTOCHRONE PROBLEM. (*See Sections* 4, 15, *and* 18.) Given two points in a vertical plane, find the curve joining them down which a particle without friction and without initial velocity will slide in the shortest time.

SOLUTION: We wish to minimize the functional

$$J[y] = \int_0^l \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\, dx, \qquad [y(0) = 0, y(l) = h].$$

A first integral of the Euler equation is

$$y(1 + y'^2) = A$$

which implies that

$$y' = \sqrt{\frac{A - y}{y}}.$$

We set $y = A \sin^2 (t/2)$ in which t is a parameter. Then

$$A \sin \frac{t}{2} \cos \frac{t}{2} \frac{dt}{dx} = \frac{\cos (t/2)}{\sin (t/2)}$$

and hence $dx = A \sin^2 (t/2)\, dt$. Then

$$x = B + \frac{A}{2} (t - \sin t)$$

and

$$y = \frac{A}{2} (1 - \cos t).$$

The extremals are cycloids.

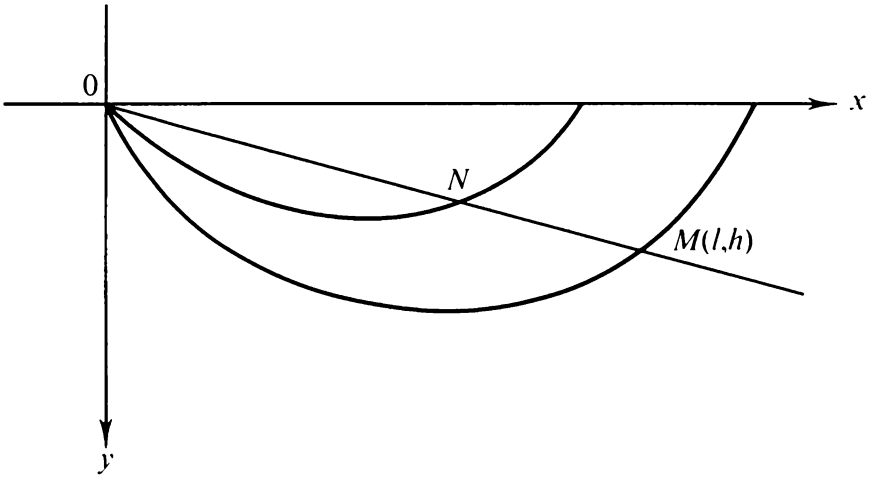

Figure 10

Let us consider the "special" cycloid

$$x = t - \sin t, \qquad y = 1 - \cos t.$$

We join the point $M(l, h)$ to O by a straight line and determine the point N, the intersection of this line with the "special" cycloid (Figure 10). The ratio OM/ON will give us the similarity coefficient that we need to obtain the unique extremal that joins O to M: $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

It is convenient to use parametric form. In this case, the existence of a field is obvious. It is represented by arcs of cycloids

$$x = C(t - \sin t) + \pi(a - C)$$

$$y = C(1 - \cos t), \qquad (0 \leq t \leq 2\pi)$$

in which C assumes any positive value. The Weierstrass condition does not need to be tested. There will be a strong minimum.

A–18. (*See Section* 19.) A surface is called a *Liouville surface* if the differential of arc on it can be expressed in the form

$$ds = \sqrt{\phi(u) + \psi(v)}\,\sqrt{du^2 + dv^2}. \tag{1}$$

We wish to find the equations of geodesic lines on a Liouville surface.

SOLUTION: The problem is to find extremals for the functional

$$J[v] = \int \sqrt{\phi(u) + \psi(v)}\,\sqrt{1 + v'^2}\,du.$$

The Hamilton-Jacobi equation has the form

$$\left(\frac{\partial W}{\partial u}\right)^2 + \left(\frac{\partial W}{\partial v}\right)^2 = \phi(u) + \psi(v).$$

We are seeking an integral in the form

$$W = W_1(u) + W_2(v).$$

This yields

$$W_1'(u) = \sqrt{\phi(u) + \beta},$$

$$W_2'(v) = \sqrt{\psi(v) - \beta}.$$

Hence

$$W = \int \sqrt{\phi(u) + \beta}\,du + \int \sqrt{\psi(v) - \beta}\,dv$$

and to find the extremals we construct the equation

$$\frac{\partial W}{\partial \beta} = \alpha.$$

It yields

$$\int \frac{du}{\sqrt{\phi(u) + \beta}} - \int \frac{dv}{\sqrt{\psi(v) - \beta}} = \alpha.$$

A simple case of a Liouville surface is a surface of revolution; its equation in cylindrical coordinates is

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad z = g(r).$$

Hence

$$\begin{aligned} ds &= \sqrt{dr^2 + r^2\, d\theta^2 + [g'(r)]^2\, dr^2} \\ &= r \sqrt{d\theta^2 + \frac{1 + [g'(r)]^2}{r^2}\, dr^2} \end{aligned} \tag{2}$$

and if we introduce a new curvilinear coordinate

$$u = \int \frac{\sqrt{1 + [g'(r)]^2}}{r}\, dr$$

so that

$$r = F(u)$$

equation (2) becomes

$$ds = F(u)\sqrt{du^2 + dv^2}$$

which is a simple case of equation (1).

Geodesic lines on a surface of revolution are extremals for the functional

$$\int \sqrt{r^2 + [1 + g'(r)]^2 r'^2}\, d\theta.$$

A first integral of the Euler equation can be found here at once:

$$\frac{r^2}{\sqrt{r^2 + \{1 + [g'(r)]^2\} r'^2}} = \text{const}$$

or

$$r^2 \frac{d\theta}{ds} = \text{const.}$$

This relation is expressed by the following theorem of Clairaut.

At every point of an arbitrary geodesic on a surface of revolution the radius of a circle of latitude multiplied by the sine of the angle between the geodesic and the meridian (Figure 11) is constant:

$$r \sin \omega = C.$$

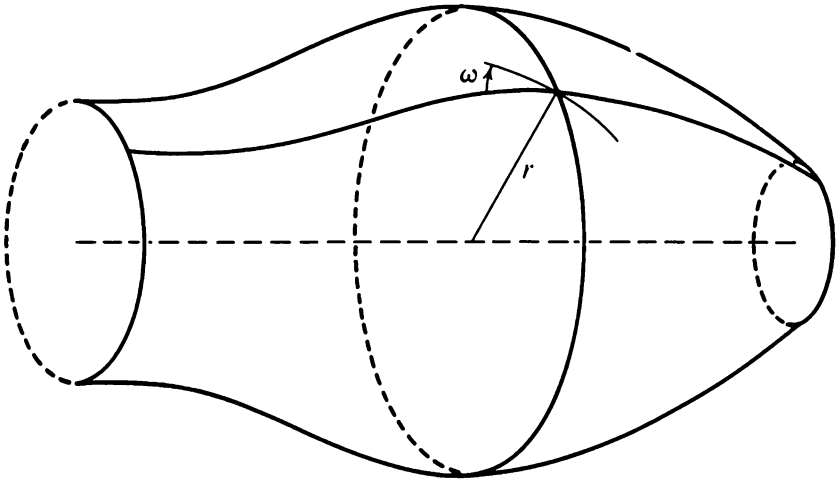

Figure 11

A–19. (*See Section* 10.) We wish to find geodesic lines in n-dimensional space with metric

$$ds^2 = \sum_{i,k=1}^{n} g_{ik}\, dx_i\, dx_k, \qquad (g_{ik} = g_{ki}).$$

SOLUTION: The problem is to find extremals for the functional in parametric form

$$J_C = \int_{t_1}^{t_2} \sqrt{\sum_{i,k=1}^{n} g_{ik}\dot{x}_i\dot{x}_k}\, dt.$$

We have n Euler equations

$$\frac{1}{2}\frac{\sum_{j,k=1}^{n}\frac{\partial g_{jk}}{\partial x_i}\dot{x}_j\dot{x}_k}{\sqrt{\sum_{j,k=1}^{n}g_{jk}\dot{x}_j\dot{x}_k}} = \frac{d}{dt}\frac{\sum_{k=1}^{n}g_{ik}\dot{x}_k}{\sqrt{\sum_{j,k=1}^{n}g_{jk}\dot{x}_j\dot{x}_k}}, \qquad (i = 1, 2, \cdots n). \quad (1)$$

These equations are not independent; one is a consequence of the others.* Hence we can impose the condition

$$\sum_{i,k=1}^{n} g_{ik}\dot{x}_i\dot{x}_k = 1$$

which expresses the fact that arc length has been chosen as parameter. Then equation (1) becomes

$$\frac{d}{ds}\sum_{k=1}^{n} g_{ik}\frac{dx_k}{ds} - \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial g_{jk}}{\partial x_i}\frac{dx_j}{ds}\frac{dx_k}{ds} = 0, \qquad (i = 1, 2, \cdots, n).$$

We introduce the Christoffel symbols

$$\begin{bmatrix} jk \\ i \end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_i}\right)$$

and

$$\begin{Bmatrix} jk \\ m \end{Bmatrix} = \sum_{i=1}^{n} g^{im}\begin{bmatrix} jk \\ i \end{bmatrix}.$$

Here the quantities g^{rs} are defined by the equations

$$\sum_{s=1}^{n} g_{rs}g^{st} = \delta_r^t = \begin{cases} 1 & (r = t) \\ 0 & (r \neq t). \end{cases}$$

Then we can express the differential equations of the geodesics in the form

$$\frac{d^2x_m}{ds^2} + \sum_{j,k=1}^{n}\begin{Bmatrix} jk \\ m \end{Bmatrix}\frac{dx_j}{ds}\frac{dx_k}{ds} = 0 \qquad (m = 1, 2, \cdots, n).$$

*This is a simple case of a general fact that follows from the positive homogeneity of the integrand function relative to the variables $\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n$. This was mentioned for $n = 2$ in Section 10. In the general case it can be shown just as simply. We suggest that the reader derive this result with the help of formula (5) Section 4 and its corollary stated at the end of Section 4.

A–20. VARIATIONAL PRINCIPLES OF MECHANICS. (*See Sections* 4 *and* 10.) We consider a mechanical system all possible positions of which are defined by the values of n independent generalized coordinates $q_1, q_2, \cdots, q_n$. Suppose forces acting on the system have a force function U which may even depend on the time, t. The kinetic energy will be denoted by T; this is a polynomial of second degree in the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, whose coefficients are functions of the generalized coordinates and possibly time.

The equations of motion, as is well known in mechanics, can be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad (i = 1, 2, \cdots, n) \tag{1}$$

where

$$L = T + U$$

is the so-called *Lagrange function* for the system considered.

But equations (1) are the Euler equations for the functional

$$\int_{t_1}^{t_2} L\, dt. \tag{2}$$

Thus we have the *Hamilton-Ostrogradski principle of stationary action.**

If the position of a system is given at an initial time t_1 and at a final time t_2 then the equation

$$\delta \int_{t_1}^{t_2} (T + U)\, dt = 0$$

determines the motion of the system; in other words, the functions

$$q_i = \bar{q}_i(t), \qquad (t_1 \leq t \leq t_2), \qquad (i = 1, 2, \cdots, n)$$

that describe the motion of the system from a given initial position to a final position furnish the integral (2) *with a stationary value.*

This principle† can be applied to a dynamic system. Because

*This is commonly known as Hamilton's principle.

†Maupertuis announced this principle without proof and Lagrange then generalized it. Jacobi formulated it rigorously and proved it in his lectures on dynamics. The following excerpt from the beginning of his sixth lecture is

only the kinetic energy and the force function are involved in the formulation of the Hamilton principle it can easily be extended to mechanical systems with infinitely many degrees of freedom; thus it is suitable for the mechanics of continuous media. This principle is also applicable to other parts of theoretical physics, both classical and relativistic.

Another important but less general variation principle of dynamics is the Maupertuis-Lagrange principle. It applies only to problems in which the function U does not contain time explicitly and the kinetic energy T is a quadratic form in the generalized velocities with coefficients that depend only on the generalized coordinates:

$$U = U(q_1, q_2, \cdots, q_n), \qquad T = \sum_{i,k=1}^{n} a_{ik}\dot{q}_i\dot{q}_k, \qquad (a_{ik} = a_{ki}).$$

In this case the equations of motion (1) have a first integral

$$L - \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = -h$$

in which h is a constant. It is not difficult to express this equation in the form

$$T - U = h \tag{3}$$

which is known as the *energy integral* because it states the fact that the energy of the system is constant.

The energy integral provides the relation

$$dt = \frac{\sqrt{\sum_{i,k=1}^{n} a_{ik}\, dq_i\, dq_k}}{\sqrt{U + h}}. \tag{4}$$

If the trajectories of the motion of the system are known in parametric form

$$q_i = q_i(\tau), \qquad (i = 1, 2, \cdots, n)$$

---

quite curious: "In almost all textbooks, even in the better ones of Poisson, Lagrange, and Laplace, this principle has been presented, in my opinion, in such a way that it can not be understood." See K. Jacobi, "Lectures on dynamics," ONTI, second edition, 1936, p. 39.

then relation (4) allows us to connect the parameter τ with time and thus to determine the law of motion.

The principle of Maupertuis-Lagrange is this: *For the motion of a system with given energy constant h the following functional is stationary,*

$$2\int \sqrt{U+h}\sqrt{\sum_{i,k=1}^{n} a_{ik}\,dq_i\,dq_k} = 2\int \sqrt{U+h}\sqrt{\sum_{i,k=1}^{n} a_{ik}\dot{q}_i\dot{q}_k}\,d\tau. \quad (5)$$

Whereas integral (2) is a functional in ordinary form, the integral (5) is a functional in parametric form. This functional can, of course, be written in the form

$$2\int T\,dt$$

which is called the *Lagrange action.* However, we must remember that time must be eliminated from the integrand function by means of the energy integral (3). This circumstance which formerly was not stressed sufficiently has led to various fallacies.

To verify the Maupertuis-Lagrange principle we must derive the Euler equations for the functional (5) and use the energy integral (3) or relation (4) to show that these Euler equations are the equations of dynamics; that is, they are the Lagrange equations of the second kind.

The Euler equations for the functional (5) have the form

$$2\frac{d}{d\tau}\sqrt{U+h}\,\frac{\sum_{k=1}^{n} a_{ik}\frac{dq_k}{d\tau}}{\sqrt{\sum_{j,k=1}^{n} a_{jk}\frac{dq_j}{d\tau}\frac{dq_k}{d\tau}}} - \frac{\sqrt{\sum_{j,k=1}^{n} a_{jk}\frac{dq_j}{d\tau}\frac{dq_k}{d\tau}}}{\sqrt{U+h}}\frac{\partial U}{\partial q_i}$$

$$-\frac{\sqrt{U+h}}{\sqrt{\sum_{j,k=1}^{n} a_{jk}\frac{dq_j}{d\tau}\frac{dq_k}{d\tau}}}\sum_{j,k=1}^{n}\frac{\partial a_{jk}}{\partial q_i}\frac{dq_j}{d\tau}\frac{dq_k}{d\tau} = 0, \qquad (i = 1, 2, \cdots, n)$$

They are not independent; one of them is a consequence of the others. But if we use equation (4) to replace the parameter τ by time t, then we obtain independent equations for the definition of the coordinates as functions of time. These equations have the form

$$2\frac{d}{dt}\sum_{k=1}^{n} a_{ik}\frac{dq_k}{dt} - \frac{\partial U}{\partial q_i} - \sum_{j,k=1}^{n}\frac{\partial a_{jk}}{\partial q_i}\frac{dq_j}{dt}\frac{dq_k}{dt} = 0, \qquad (i = 1, 2, \cdots, n)$$

They are easily expressed in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial U}{\partial q_i} - \frac{\partial T}{\partial q_i} = 0, \qquad (i = 1, 2, \cdots, n)$$

and are really the Lagrange equations of the second kind.

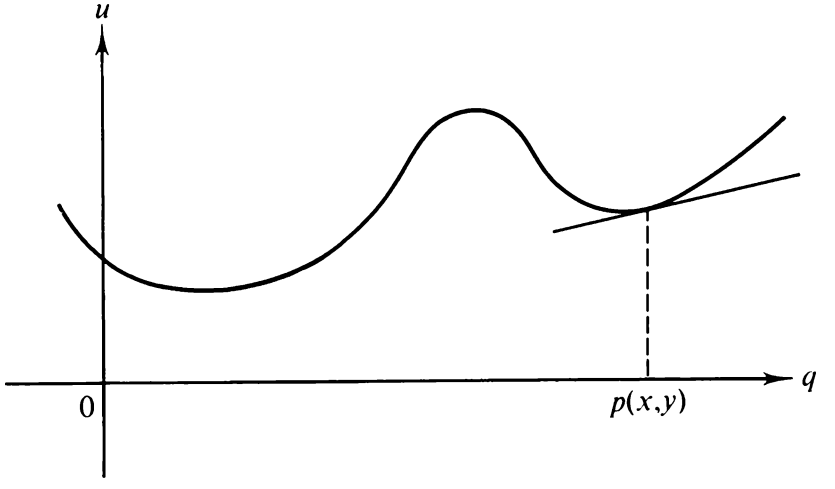

Figure 12

A–21. (*See Section* 14.) For every point $(x, y) \in G$ we can construct in the qu-plane a curve (Figure 12) with equation

$$u = f(x, y, q), \qquad (-\infty < q < \infty)$$

where $f(x, y, y')$ is the integrand function of the functional we are considering. This curve is called the *indicatrix*. In the case of a regular functional the indicatrix is concave in the direction of positive u for any point $(x, y) \in G$.

The Weierstrass condition

$$E(x, y, p, q) \geq 0$$

in which $p = p(x, y)$ is the slope function of the field at the point considered shows that the indicatrix constructed for an arbitrary point (x, y) lies above its tangent at the point with abscissa $p(x, y)$.

The equation of the tangent to the indicatrix at a point with abscissa $p(x, y)$ is

$$u - f(x, y, p) = (q - p)f_{y'}(x, y, p)$$

and the point $(q, f(x, y, q))$ lies above this tangent if and only if

$$f(x, y, q) - f(x, y, p) - (q - p)f_{y'}(x, y, p) \geq 0.$$

The strengthened Legendre condition

$$f_{y'y'}(x, \bar{y}(x), \bar{y}'(x)) > 0$$

shows that for every point of the extremal $(x, \bar{y}(x))$ the indicatrix constructed for this point is concave upward at a point of the indicatrix for which $q = \bar{y}'(x)$.

A point $(x_0, y_0) \in G$ can be a corner point of the extremal if the indicatrix for this point has a double tangent, that is, if there exists a straight line tangent to the indicatrix at two different points. For this the abscissas of these points of tangency must be the slopes of the extremal at the corner point.

A–22. (*See Sections* 4 *and* 15.) We wish to find a minimum for the functional

$$J[y] = \int_0^\alpha (y'^4 - 6y'^2)\, dx \qquad [y(0) = 0, y(\alpha) = \beta].$$

SOLUTION: The extremals are straight lines. Suppose the straight line joining the points $(0, 0)$ and (α, β) has slope m. The straight lines $y = mx +$ const constitute a field. The Weierstrass function is

$$E(x, y, m, q) = (q - m)^2\{q^2 + 2mq + (3m^2 - 6)\}.$$

If $m^2 \geq 3$, there is a strong minimum. If $m^2 < 3$ a straight line will not furnish a strong extremum. Furthermore, because

$$f_{y'y'}(x, y, y') = 12(y'^2 - 1)$$

a straight line will furnish a weak minimum if $1 < m^2 < 3$ and a weak maximum if $m^2 < 1$.

Figure 13 shows regions for the point (α, β) that illustrate the various possibilities: regions I correspond to a strong minimum,

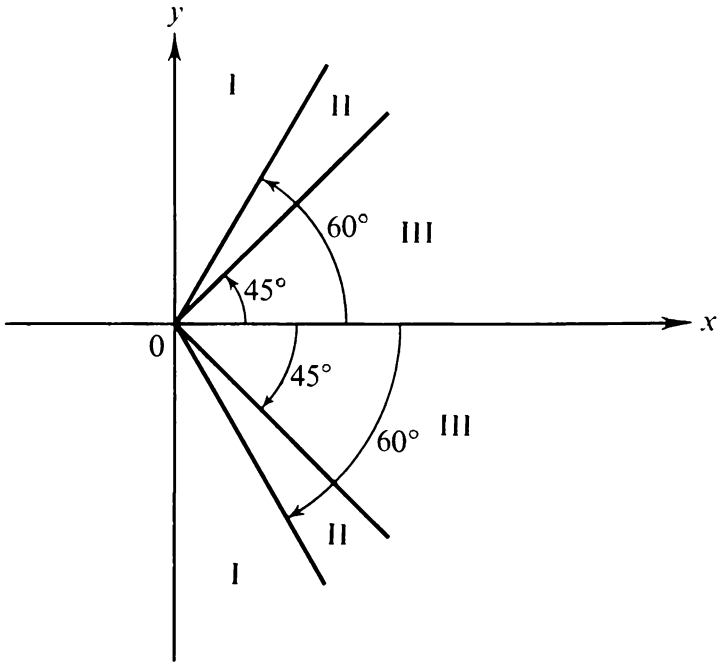

Figure 13

regions II a weak minimum, and regions III a weak maximum. Let us consider broken extremals. First we notice that the indicatrix does not depend on the point (x, y). We have a double tangent that touches the indicatrix at points with abscissas $\pm\sqrt{3}$, see Figure 14. Hence the Weierstrass-Erdmann condition will be satisfied if we use broken extremals that make angles of $\pm 60°$ with the x-axis. For example, if $\alpha = 2$ and $\beta = 0$

$$J[y] = \int_0^2 (y'^4 - 6y'^2)\, dx, \qquad [y(0) = 0, y(2) = 0].$$

On the broken line y_1 with one corner point (Figure 15) the functional has the value

$$J[y_1] = -18.$$

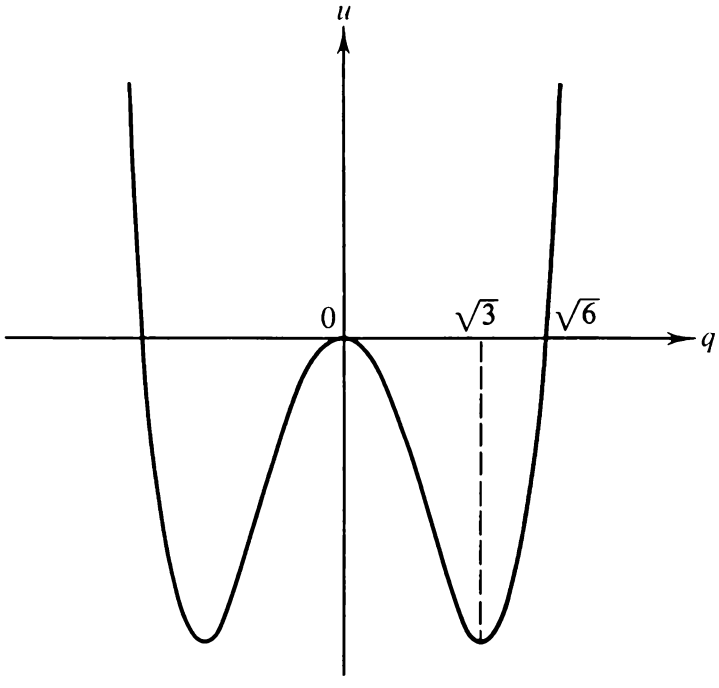

Figure 14

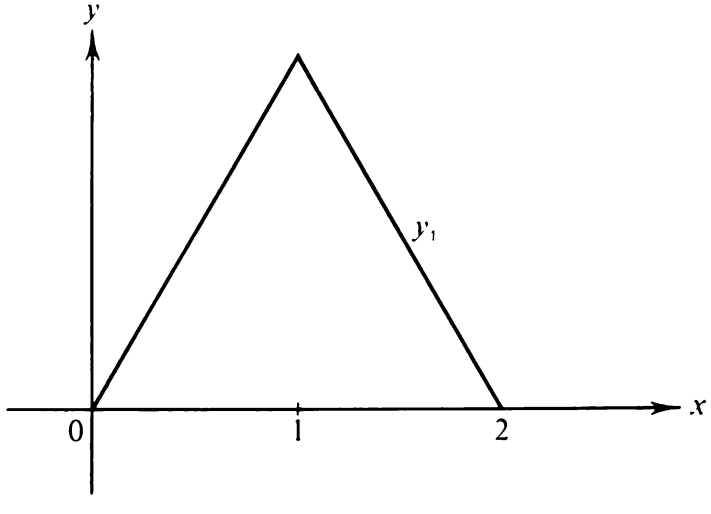

Figure 15

$J[y]$ will have the same value on the broken line y_2 with two corners (Figure 16) and also on the broken line y_3 with three corners (Figure 17), etc.

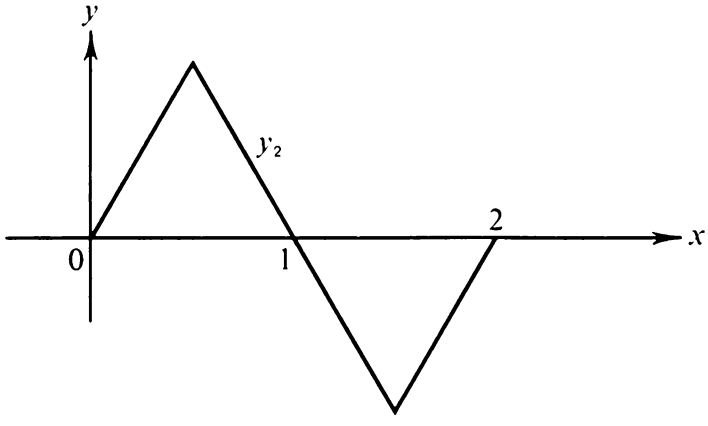

Figure 16

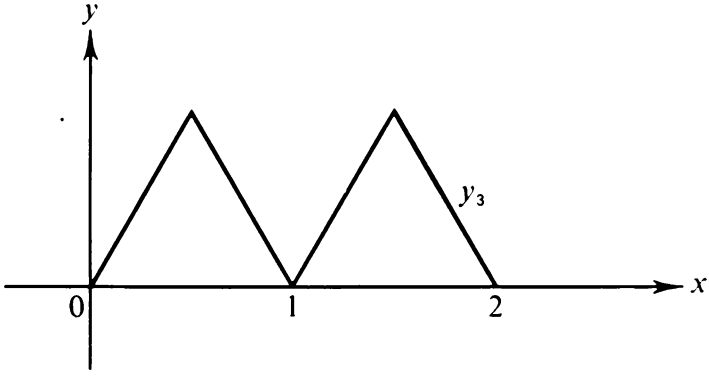

Figure 17

We can verify that each of these broken lines will furnish a strong minimum by considering the increment. For example, if we set $y = y_1 + \omega$ we find:

$$J[y] - J[y_1]$$

$$= \int_0^2 \{4y_1'^3\omega' + 6y_1'^2\omega'^2 + 4y_1'\omega'^3 + \omega'^4 - 12y_1'\omega' - 6\omega'^2\}\, dx$$

$$= \int_0^1 (12\omega'^2 + 4\sqrt{3}\,\omega'^3 + \omega'^4)dx + \int_1^2 (12\omega'^2 - 4\sqrt{3}\,\omega'^3 + \omega'^4)dx$$

$$= \int_0^1 \omega'^2(\omega' + 2\sqrt{3})^2\, dx + \int_1^2 \omega'^2(\omega' - 2\sqrt{3})^2\, dx \geq 0.$$

A–23. (*See Sections* 22 *and* 27.) For the simplest functional

$$\int_a^b f(x, y, y')\, dx$$

there always exist admissible functions (that is, piecewise-smooth functions that satisfy the given boundary conditions) for which it is defined and has a finite value. For functionals

$$\iint_D f(x, y, u, u_x, u_y)\, d\sigma$$

there may not exist one admissible function $u(x, y)$ that furnishes the functional with a finite value. The first example of this kind is due to Hadamard and concerns the Dirichlet integral

$$\iint_D (u_x^2 + u_y^2)\, d\sigma.$$

The Euler equation for this functional is the equation of logarithmic potential; consequently, the extremals for the Dirichlet integral are harmonic functions.

Let us suppose that the region D is a circle $x^2 + y^2 < \rho^2$ and denote this circle by K_ρ and the Dirichlet integral for this circle by D_ρ:

$$D_\rho[u] = \iint_{K_\rho} (u_x^2 + u_y^2)\, d\sigma.$$

In polar coordinates the Dirichlet integral becomes

$$D_\rho[u] = \iint_{K_\rho} \left(u_r^2 + \frac{1}{r^2}\, u_\theta^2\right) r\, dr\, d\theta.$$

For any continuous function $g = g_P$ of the point P on the circumference of the circle K_1 there exists a function, U harmonic in K_1 (that is, an extremal) that takes the value g_P on the boundary of the circle K_1. To construct this function it is better to use polar coordinates; then if

$$g = g(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta)$$

it follows that

$$U = U(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta), \qquad (0 \leq r < 1). \quad (1)$$

We shall show that the Dirichlet integral for the function $U = U(r, \theta)$ is not always finite. After this we shall show that if the Dirichlet integral has an infinite value for this function $U(r, \theta)$ then it will not be finite for any admissible function.* To prove that the Dirichlet integral for $U = U(r, \theta)$ is not always finite let us take the function

$$\tilde{g} = \tilde{g}(\theta) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos (k!\theta).$$

The corresponding harmonic function is equal to

$$\tilde{U} = \tilde{U}(r, \theta) = \sum_{k=1}^{\infty} \frac{r^{k!}}{k^2} \cos (k!\theta).$$

For arbitrary positive $R < 1$ we have

$$\begin{aligned} D_R[\tilde{U}] &= \int_0^R r\, dr \int_0^{2\pi} \left\{ \tilde{U}_r^2 + \frac{1}{r^2} \tilde{U}_\theta^2 \right\} d\theta \\ &= 2\pi \int_0^R r \sum_{k=1}^{\infty} \frac{(k!)^2}{k^4} r^{2(k!-1)}\, dr \\ &= \pi \sum_{k=1}^{\infty} \frac{k!}{k^4} R^{2(k!)}. \end{aligned}$$

*An admissible function is one that is piecewise-smooth in the closed circle K_1 and that coincides with the function g on the circumference of this circle.

From this it follows that

$$D_1[\tilde{U}] = \lim_{R \to 1-0} D_R[\tilde{U}] = \infty.$$

To prove that the Dirichlet integral will not be finite for any admissible function if it has an infinite value for $U(r, \theta)$ we suppose that the Dirichlet integral has a finite value for some piecewise-smooth function $V(r, \theta)$. Consider the harmonic function $U(r, \theta)$, defined by formula (1), that assumes the values $g = g(\theta)$ on the circumference of the circle K; the function $V(r, \theta)$ also assumes the values $g = g(\theta)$. We note that

$$a_k = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos k\theta \, d\theta, \qquad b_k = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin k\theta \, d\theta.$$

We also set

$$U_n(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{n} r^k (a_k \cos k\theta + b_k \sin k\theta)$$

and

$$W_n(r, \theta) = V(r, \theta) - U_n(r, \theta).$$

Designating the quantity

$$\lim_{r \to 1-0} W_n(r, \theta) = g(\theta) - U_n(1, \theta)$$

by $W_n(1, \theta)$ we can write the equation

$$\int_0^{2\pi} W_n(1, \theta) \cos k\theta \, d\theta = \int_0^{2\pi} W_n(1, \theta) \sin k\theta \, d\theta = 0,$$
$$(k = 0, 1, 2, \cdots, n). \quad (2)$$

Now we notice that for arbitrary positive $R < 1$

$$D_R[V] = D_R[U_n] + D_R[W_n] + 2 \iint_{K_R} \left(\frac{\partial U_n}{\partial x} \frac{\partial W_n}{\partial x} + \frac{\partial U_n}{\partial y} \frac{\partial W_n}{\partial y} \right) d\sigma.$$

The last term on the right side approaches zero as $R \to 1$. Indeed,

$$\omega_R \equiv \iint_{K_R} \left(\frac{\partial U_n}{\partial x} \frac{\partial W_n}{\partial x} + \frac{\partial U_n}{\partial y} \frac{\partial W_n}{\partial y} \right) d\sigma = \iint_{K_R} \operatorname{div} (W_n \operatorname{grad} U_n) \, d\sigma$$

because U_n is a harmonic function and consequently

$$\iint_{K_R} W_n \, \Delta U_n \, d\sigma = 0.$$

Hence

$$\omega_R = \oint W_n \frac{\partial U_n}{\partial r} \, ds = \int_0^{2\pi} W_n(R, \theta) \frac{\partial U_n(R, \theta)}{\partial R} \, R \, d\theta$$

$$= \sum_{k=1}^{n} kR^k \int_0^{2\pi} W_n(R, \theta)(a_k \cos k\theta + b_k \sin k\theta) \, d\theta$$

from which

$$\lim_{R \to 1} \omega_R = \sum_{k=1}^{k} k \int_0^{2\pi} W_n(1, \theta)(a_k \cos k\theta + b_k \sin k\theta) \, d\theta.$$

Equation (2) implies that the last integral is equal to zero. Thus

$$D_1[V] = D_1[U_n] + D_1[W_n]$$

which implies that

$$D_1[V] \geq D_1[U_n].$$

This inequality allows us to write

$$D_1[V] \geq D_R[U_n]$$

for any positive $R < 1$. Proceeding to the limit we have

$$D_1[V] \geq D_R[U].$$

The second limit gives

$$D_1[V] \geq D_1[U].$$

This shows that $D_1[U] < \infty$. Thus, if $D_1[U] = \infty$ then $D_1[V] = \infty$ for any admissible function V. Our assertion has been proved.

A–24. GENERALIZED LEMMA OF THE CALCULUS OF VARIATIONS. (*See Sections* 3, 5, *and* 22–24.) In Section 23, to derive the Euler equation, we used the proposition: *If $M(x)$ is a piecewise-continuous function and if*

$$\int_a^b M(x)\eta^{(n)}(x) \, dx = 0$$

for every function $\eta(x)$ that has a piecewise-continuous derivative of order n, and that vanishes with its derivatives of all orders up to and including order $n-1$ at the end points of the interval $[a, b]$, then there exist constants $c_0, c_1, \cdots, c_{n-1}$ such that

$$M(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}.$$

That is, $M(x)$ is a polynomial of degree $n-1$. This proposition is a particular case of

THE GENERALIZED LEMMA OF THE CALCULUS OF VARIATIONS: *If*

$$\int_a^b M(x)\zeta(x)\,dx = 0 \tag{1}$$

and the conditions

$$\int_a^b \zeta(x)\phi_i(x)\,dx = 0, \qquad (i = 1, 2, \cdots, n) \tag{2}$$

are satisfied in which the $\phi_i(x)$, $(i = 1, 2, \cdots, n)$ are given functions, then almost everywhere

$$M(x) = \sum_{i=1}^{n} c_i\phi_i(x).$$

PROOF: Equation (1) expresses the fact that the function $M(x)$ is orthogonal to the function $\zeta(x)$ and equation (2) that the function $\zeta(x)$ is orthogonal to the functions $\phi_i(x)$, $(i = 1, 2, \cdots, n)$. We can assume that the functions are linearly independent. Then we can determine the constants c_i uniquely so that the difference

$$M(x) - \sum_{i=1}^{n} c_i\phi_i(x) = \zeta_0(x)$$

will be orthogonal to all the functions $\phi_i(x)$, $(i = 1, 2, \cdots, n)$. To do this we must solve the following system for c_i

$$c_1(\phi_1, \phi_i) + c_2(\phi_2, \phi_i) + \cdots + c_n(\phi_n, \phi_i) = (M - \zeta_0, \phi_i), \qquad (i = 1, 2, \cdots, n)$$

the determinant of which is the Gram determinant of the functions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$; it is not zero. Thus the function $\zeta_0(x)$ satisfies conditions (2) and it can thus be used as $\zeta(x)$ in relation

(1). Then we have

$$\int_a^b \left\{M(x) - \sum_{i=1}^{n} c_i\phi_i(x)\right\}^2 = \int_a^b M(x)\zeta_0(x)\,dx - \sum_{i=1}^{n} c_i \int_a^b \phi_i(x)\zeta_0(x)\,dx = 0.$$

From this equation it follows that almost everywhere

$$M(x) - \sum_{i=1}^{n} c_i\phi_i(x) = 0.$$

A–25. KELVIN'S PROBLEM. (*See Section* 25.) A mass with continuous density $\mu(x, y)$ covers the xy-plane; a piecewise-smooth curve C and two points on it P_1, P_2 are given (Figure 18). Among

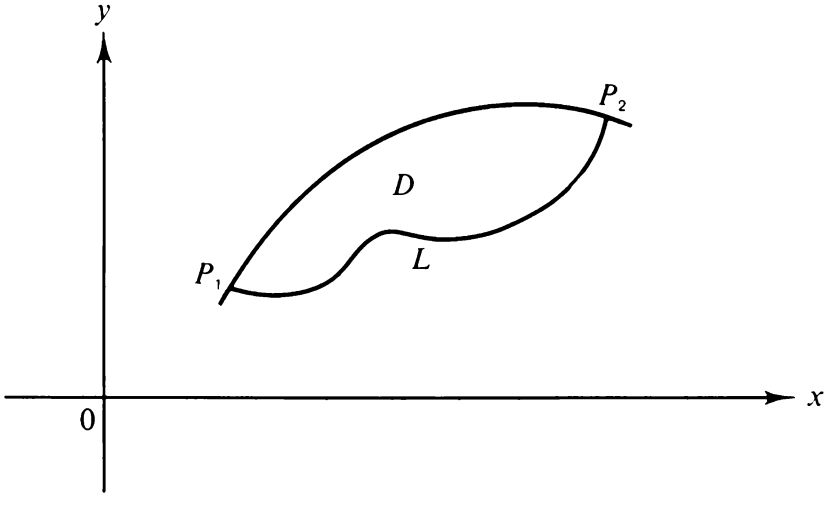

Figure 18

all curves of a given length L that join P_1 and P_2, find the one that, with the arc P_1P_2 of the curve C, bounds a region of maximum mass. The points P_1 and P_2 may coincide.

SOLUTION: Consider the function

$$V(x, y) = \int \mu(x, y)\,dx.$$

Then

$$\iint_D \mu(x, y)\, dx\, dy = \iint_D \frac{\partial V}{\partial x}\, dx\, dy = \oint V\, dy.$$

The term on the right is the sum of two integrals: one taken along the curve L and the other taken along the arc P_2P_1 of the curve C and having a known value that we designate by K. Hence

$$\iint_D \mu(x, y)\, dx\, dy = \int_{t_1}^{t_2} V(x, y)\dot{y}\, dt + K$$

and the problem is to find the maximum of the functional

$$\int_{t_1}^{t_2} V(x, y)\dot{y}\, dt$$

for the condition that

$$\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = L.$$

Consider the auxiliary function

$$F = V\dot{y} - \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$$

and write the Euler equation in Weierstrass form:

$$\frac{1}{\rho} - \frac{1}{\lambda}\frac{\partial V}{\partial x} = 0 \tag{1}$$

where ρ is the radius of curvature of the desired curve. The definition of V allows us to express equation (1) as follows:

$$\frac{1}{\rho} = \frac{\mu(x, y)}{\lambda}. \tag{2}$$

In the particular case in which μ = const, we have the isoperimetric problem in the narrow sense and it follows from equation (2) that the extremals are circles.

A–26. (*See Section* 25.) The isoperimetric problem in the narrow sense admits various methods of solution. First we show that it

can be reduced to a problem without a side condition; and then we show that it can be solved by a particular application of trigonometric series.

A. We note that the length of a closed curve will not vary if we replace an arc of it by another arc symmetric to its chord. Hence the desired curve must be convex and if two points divide it into two arcs of equal length, the areas of the figures bounded by each of these arcs and their common chord must be identical. Thus the isoperimetric problem reduces to this: of all arcs of given length $\frac{1}{2}L$ that lie in the upper half plane and have their end points on the x-axis, if one of the end points is given (say, $x = 0$) find the curve that encloses the greatest area with the x-axis.

Admissible curves are given by the equations

$$x = x(s), \qquad y = y(s), \qquad (0 \leq s \leq \tfrac{1}{2}L)$$

in which the functions $x(s), y(s)$ are connected by the relation

$$x'^2 + y'^2 = 1$$

and satisfy the conditions

$$x(0) = 0, \qquad y(0) = 0, \qquad y(\tfrac{1}{2}L) = 0.$$

The functional to be maximized is

$$J = \int_0^{\frac{1}{2}L} y(s)x'(s)\, ds.$$

We have before us a typical Lagrange problem. However, if we use the side equation we can easily find $x'(s)$ so that the functional becomes

$$J[y] = \int_0^{\frac{1}{2}L} y(s)\sqrt{1 - [y'(s)]^2}\, ds \tag{1}$$

with boundary conditions

$$y(0) = y(\tfrac{1}{2}L) = 0. \tag{2}$$

We have reached the simplest problem of the calculus of variations. We stress the fact that the form of the function

$$f = y\sqrt{1 - z^2}$$

determines the restriction

$$-1 \leq z \leq 1$$

on z. This fact does not prevent the application of the general theory because we supposed for simplicity that z assumed arbitrary finite values.

The Euler equation has a first integral

$$y = C\sqrt{1 - y'^2}$$

and hence

$$y = C \sin \frac{s - s_0}{C}.$$

Boundary conditions (2) and the requirement that $y \geq 0$ allow us to define the arbitrary constants s_0, C and we find

$$y = \frac{L}{2\pi} \sin \frac{2\pi s}{L}. \tag{3}$$

With the aid of the side condition and the condition

$$x(0) = 0$$

x is defined in the form

$$x = \frac{L}{2\pi}\left\{1 - \cos \frac{2\pi s}{L}\right\}.$$

We can apply general criteria to show that the function (3) provides the functional (1) with a strong minimum.

B. The equation of every closed curve having length L can be expressed in the form

$$x = a_0 + \sum_1^\infty \left(a_k \cos \frac{2k\pi s}{L} + c_k \sin \frac{2k\pi s}{L}\right),$$

$$y = b_0 + \sum_1^\infty \left(b_k \cos \frac{2k\pi s}{L} + d_k \sin \frac{2k\pi s}{L}\right)$$

where s is the arc length of the curve measured from some point. The absolute and uniform convergence of these series follows from the fact that the functions $x = x(s), y = y(s)$ satisfy a Cauchy-

Lipschitz condition:

$$|x(s'') - x(s')| \leq |s'' - s'|,$$
$$|y(s'') - y(s')| \leq |s'' - s'|.$$

It also follows from these inequalities that the derivatives dx/ds, dy/ds exist almost everywhere and do not exceed unity numerically. Hence these derivatives and their squares are integrable. Consequently,

$$\frac{dx}{ds} \sim \frac{2\pi}{L} \sum_1^\infty k \left(c_k \cos \frac{2k\pi s}{L} - a_k \sin \frac{2k\pi s}{L} \right),$$
$$\frac{dy}{ds} \sim \frac{2\pi}{L} \sum_1^\infty k \left(d_k \cos \frac{2k\pi s}{L} - b_k \sin \frac{2k\pi s}{L} \right)$$

and the Parseval-Lyapunov equation applies to all four functions x, y, x', y'. This yields the following expressions for the length of the curve L and the area S that it bounds:

$$L = \int_0^L \left\{ \left(\frac{dx}{ds} \right)^2 + \left(\frac{dy}{ds} \right)^2 \right\} ds$$
$$= \frac{2\pi^2}{L} \sum_1^\infty k^2 (a_k^2 + c_k^2 + b_k^2 + d_k^2), \tag{4a}$$

$$S = \frac{1}{2} \int_0^L \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right) ds = \pi \sum_1^\infty k(a_k d_k - b_k c_k). \tag{4b}$$

It follows that from these formulas

$$L^2 - 4\pi S$$
$$= 2\pi^2 \sum_1^\infty \{ (k^2 - 1)(b_k^2 + d_k^2) + (ka_k - d_k)^2 + (kc_k + b_k)^2 \} \geq 0.$$

Thus for an arbitrary closed curve of length L

$$S \leq \frac{1}{4\pi} L^2.$$

The case of equality provides the curve of length L that bounds

the greatest area. This occurs if and only if

$$d_k = b_k = 0, \qquad (k = 2, 3, \cdots)$$

$$ka_k = d_k, \qquad kc_k = -b_k, \qquad (k = 1, 2, \cdots)$$

that is, only for

$$\begin{aligned} x &= a_0 + a_1 \cos \frac{2\pi s}{L} - b_1 \sin \frac{2\pi s}{L}, \\ y &= b_0 + b_1 \cos \frac{2\pi s}{L} + a_1 \sin \frac{2\pi s}{L}. \end{aligned} \tag{5}$$

Here, equations (4a) and (4b) imply that

$$L^2 = 4\pi^2(a_1^2 + b_1^2).$$

But from equation (5) it follows that

$$(x - a_0)^2 + (y - b_0)^2 = a_1^2 + b_1^2$$

so that the extremal curve is a circle of radius

$$\sqrt{a_1^2 + b_1^2} = \frac{L}{2\pi}.$$

A–27. (*See Section* 26.) Suppose that

$$\Phi(x, y, z) = 0 \tag{1}$$

is the equation of a smooth surface. We wish to show that along each of its geodesics, the principal normal coincides with the normal to the surface.

SOLUTION: Suppose that along some arc of a geodesic line

$$\frac{dx}{ds} \neq 0. \tag{2}$$

This arc is then an extremal for the functional

$$\int \sqrt{1 + y'^2 + z'^2}\, dx$$

with side condition (1); the Lagrange multiplier rule provides that

it is an extremal for the functional

$$\int \{\sqrt{1 + y'^2 + z'^2} + \lambda\Phi(x, y, z)\}\, dx$$

without any side condition; here λ is some function of x or of s. Thus along the geodesic arc under consideration

$$\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2 + z'^2}} - \lambda\frac{\partial\Phi}{\partial y} = 0, \tag{3a}$$

$$\frac{d}{dx}\frac{z'}{\sqrt{1 + y'^2 + z'^2}} - \lambda\frac{\partial\Phi}{\partial z} = 0. \tag{3b}$$

But

$$\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = \frac{dy}{ds}, \qquad \frac{z'}{\sqrt{1 + y'^2 + z'^2}} = \frac{dz}{ds}.$$

Hence equations (3a) and (3b) can be written in the form

$$\frac{d^2y}{ds^2} = \lambda\frac{dx}{ds}\frac{\partial\Phi}{\partial y}, \qquad \frac{d^2z}{ds^2} = \lambda\frac{dx}{ds}\frac{\partial\Phi}{\partial z}$$

or

$$\frac{d^2y}{ds^2} = \mu\frac{\partial\Phi}{\partial y}, \tag{4a}$$

$$\frac{d^2z}{ds^2} = \mu\frac{\partial\Phi}{\partial z} \tag{4b}$$

in which μ is a new function. From the equation of the surface and the identity

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$$

it follows that along a geodesic

$$\frac{\partial\Phi}{\partial x}\frac{dx}{ds} + \frac{\partial\Phi}{\partial y}\frac{dy}{ds} + \frac{\partial\Phi}{\partial z}\frac{dz}{ds} = 0,$$

$$\frac{d^2x}{ds^2}\frac{dx}{ds} + \frac{d^2y}{ds^2}\frac{dy}{ds} + \frac{dz^2}{ds^2}\frac{dz}{ds} = 0.$$

If we multiply the first of these equations by μ and subtract it from the second, we have

$$\left[\frac{d^2x}{ds^2} - \mu\frac{\partial\Phi}{\partial x}\right]\frac{dx}{ds} + \left[\frac{d^2y}{ds^2} - \mu\frac{\partial\Phi}{\partial y}\right]\frac{dy}{ds} + \left[\frac{d^2z}{ds^2} - \mu\frac{\partial\Phi}{\partial z}\right]\frac{dz}{ds} = 0.$$

Using relations (4) and (2), we find that along the considered geodesic we also have the relation

$$\frac{d^2x}{ds^2} = \mu\frac{\partial\Phi}{\partial x}.$$

The desired property of a geodesic has been proved.

A–28. CHAPLYGIN'S PROBLEM. (*See Section* 26.) What closed curve will be the path of the center of gravity of an airplane flying in a horizontal plane with natural velocity v_0 if it is to fly around the greatest area in a given time T if the wind has constant direction and constant velocity $a < v_0$?

SOLUTION: Let Ox be the direction of the wind. Suppose that α is the angle between the direction of the axis of the airplane and Ox, and that $x = x(t), y = y(t)$ are the coordinates of the center of gravity of the airplane. Then the components of the absolute velocity of the airplane will be

$$\frac{dx}{dt} = v_0\cos\alpha + a, \tag{1a}$$

$$\frac{dy}{dt} = v_0\sin\alpha \tag{1b}$$

and the area around which the plane flies is equal to

$$S = \frac{1}{2}\int_0^T \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt.$$

Here we have a Lagrange problem with two nonholonomic conditions.

Consider the auxiliary functional

$$\int_0^T \left\{ \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) + \lambda_1 \left[\frac{dx}{dt} - v_0 \cos\alpha - a \right] + \lambda_2 \left[\frac{dy}{dt} - v_0 \sin\alpha \right] \right\} dt$$

in which $\lambda_1(t), \lambda_2(t)$ are unknown functions of t.

The Euler equations for this are:

$$\frac{d}{dt}\left(-\frac{1}{2}y + \lambda_1 \right) - \frac{1}{2}\frac{dy}{dt} = 0, \tag{2a}$$

$$\frac{d}{dt}\left(\frac{1}{2}x + \lambda_2 \right) + \frac{1}{2}\frac{dx}{dt} = 0, \tag{2b}$$

$$-\lambda_2 v_0 \cos\alpha + \lambda_1 v_0 \sin\alpha = 0. \tag{2c}$$

Equations (2a) and (2b) are immediately integrable and we have

$$\lambda_1 = y, \qquad \lambda_2 = -x \tag{3}$$

if the arbitrary constants of integration are chosen to be zero by a suitable translation of the axes. It follows from equations (2) and (3) that $x\cos\alpha + y\sin\alpha = 0$. Hence we can set $x = r\sin\alpha$, $y = -r\cos\alpha$ and from equations (1a) and (1b) we have

$$\frac{dr}{dt} - a\sin\alpha = 0.$$

Because

$$\sin\alpha = \frac{1}{v_0}\frac{dy}{dt}$$

we find that

$$\frac{dr}{dt} = \frac{a}{v_0}\frac{dy}{dt}.$$

Integration of this equation yields

$$\sqrt{x^2 + y^2} = \frac{a}{v_0}y + C.$$

This is the equation of an ellipse with one focus at the origin. The major axis is perpendicular to the direction of the wind and the eccentricity is a/v_0.

A–29. (*See Section* 22.) For the derivation of the Euler equation for a double integral we assumed that the function $u = \bar{u}(x, y)$ had continuous derivatives of the second order. However, for single integrals we assumed only the existence of the first derivative.* A. Haar and his pupils have investigated the possibility of a more general theory for double integrals.†

We present a fundamental result of these generalizations.

THEOREM OF HAAR: *If a piecewise-smooth function* $\bar{u}(x, y)$ *furnishes the functional*

$$J[u] = \iint_D f(x, y, u_x, u_y)\, dx\, dy$$

with a stationary value for given values on the boundary of the region D, *then for an arbitrary simply connected region* $\Delta \subset D$ *with piecewise-smooth boundary* γ

$$\iint_\Delta \bar{f}_u\, dx\, dy = \int_\gamma \bar{f}_{u_x}\, dy - \bar{f}_{u_y}\, dx.‡ \tag{1}$$

In the particular case in which u does not appear explicitly in the function f, this result leads to the fact that the expression $\bar{f}_{u_x}\, dy - \bar{f}_{u_y}\, dx$ must be an exact differential of some function in every simply connected part of the region D.

It should be pointed out that Lagrange obtained his result for minimal surfaces in this form. We note that it is sufficient to prove Haar's theorem for the case in which Δ is a rectangle§ with oppo-

*We proved the existence of the second derivative for a nonsingular element.

†See A. Haar, "Über reguläre Variationsprobleme," *Acta Szeged*, Vol. 3, 1927, pp. 224–234.

‡If $\bar{f}_{u_x}$ has a continuous derivative with respect to x and $\bar{f}_{u_y}$ has a continuous derivative with respect to y, then relation (1) has the form

$$\bar{f}_u - \frac{\partial}{\partial x}\bar{f}_{u_x} - \frac{\partial}{\partial y}\bar{f}_{u_y} = 0.$$

§We proceed from one rectangle to a region consisting of a finite number of rectangles and then, by the limiting process, to an arbitrary region.

site vertices (x_0, y_0), (x_1, y_1). Setting the first variation equal to zero for this region leads to

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} \{\bar{f}_u\zeta + \bar{f}_{u_x}\zeta_x + \bar{f}_{u_y}\zeta_y\}\, dx\, dy = 0 \tag{2}$$

for arbitrary piecewise-smooth functions $\zeta(x, y)$ that vanish on the boundary of the rectangle.

In particular, we can set

$$\zeta(x, y) = \xi(x)\eta(y)$$

where

$$\xi(x_0) = \xi(x_1) = 0, \qquad \eta(y_0) = \eta(y_1) = 0.$$

Then equation (2) becomes

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} \{\bar{f}_u\xi(x)\eta(y) + \bar{f}_{u_x}\xi'(x)\eta(y) + \bar{f}_{u_y}\xi(x)\eta'(y)\}\, dx\, dy = 0$$

which, upon integration by parts, yields

$$\int_{x_0}^{x_1} \xi'(x)\, dx \int_{y_0}^{y_1} \left\{\bar{f}_{u_x}\eta(y) - \eta(y)\int_{x_0}^{x} \bar{f}_u\, dx - \eta'(y)\int_{x_0}^{x} \bar{f}_{u_y}\, dx\right\} dy = 0.$$

Setting $\eta(y)$ equal to some fixed function and making use of the fundamental result for single integrals we have

$$\int_{y_0}^{y_1} \left\{\bar{f}_{u_x}\eta(y) - \eta(y)\int_{x_0}^{x} \bar{f}_u\, dx - \eta'(y)\int_{x_0}^{x} \bar{f}_{u_y}\, dx\right\} dy = C \tag{3}$$

where C is some constant. If we set $x = x_0$ we find

$$C = \int_{y_0}^{y_1} \eta(y)(\bar{f}_{u_x})_{x=x_0}\, dy.$$

Hence equation (3) becomes

$$-\int_{y_0}^{y} \Big\{\eta(y)[\bar{f}_{u_x} - (\bar{f}_{u_x})_{x=x_0}] + \eta(y)\int_{x_0}^{x} \bar{f}_u\, dx + \eta'(y)\int_{x_0}^{x} \bar{f}_{u_y}\, dx\Big\}\, dy = 0.$$

Integration by parts yields

$$\int_{y_0}^{y_1} \eta'(y) \left\{ \int_{x_0}^{x} \bar{f}_{u_y}\, dx - \int_{y_0}^{y} \int_{x_0}^{x} \bar{f}_u\, dy\, dx + \int_{y_0}^{y} [\bar{f}_{u_x} - (\bar{f}_{u_x})_{x=x_0}]\, dy \right\} dy = 0.$$

Again from a result for single integrals

$$\int_{x_0}^{x} \bar{f}_{u_y}\, dx - \int_{y_0}^{y} \int_{x_0}^{x} \bar{f}_u\, dy\, dx + \int_{y_0}^{y} [\bar{f}_{u_x} - (\bar{f}_{u_x})_{x=x_0}]\, dy = C(x). \quad (4)$$

To determine $C(x)$ we set $y = y_0$. This gives

$$C(x) = \int_{x_0}^{x} (\bar{f}_{u_y})_{y=y_0}\, dx.$$

If we substitute this value of $C(x)$ in formula (4) and replace x by x_1 and y by y_1 we obtain equation (1).

This proof contains an implicit proof of the second part of Haar's lemma.

HAAR'S LEMMA: *Suppose P and Q are two continuous functions in a simply connected region D bounded by a piecewise-smooth contour; the integral*

$$\int_{\gamma} Q\, dx - P\, dy \quad (5)$$

will vanish for an arbitrary closed contour lying in D if and only if for every smooth function $\zeta(x, y)$ in D equal to zero on the boundary of D

$$\iint_{D} \left(P \frac{\partial \zeta}{\partial x} + Q \frac{\partial \zeta}{\partial y} \right) dx\, dy = 0. \quad (6)$$

Consider the first part of the proposition. The vanishing of the integral (5) for an arbitrary contour implies that equation (6) holds for any smooth function $\zeta(x, y)$ which vanishes on the boundary of D. In our case the boundary of D is a piecewise-smooth contour C and this can be proved very simply. If the integral (5) vanishes for an arbitrary contour γ lying in D it follows that in D

there exists a smooth function $\omega(x, y)$ for which

$$Q = \frac{\partial \omega}{\partial x}, \qquad P = -\frac{\partial \omega}{\partial y}.$$

Hence

$$\iint_D \left(P \frac{\partial \zeta}{\partial x} + Q \frac{\partial \zeta}{\partial y} \right) dx\, dy = \iint_D \frac{\partial(\omega, \zeta)}{\partial(x, y)}\, dx\, dy.$$

But we know that*

$$\iint_D \frac{\partial(\omega, \zeta)}{\partial(x, y)}\, dx\, dy = -\int_C \zeta \left(\frac{\partial \omega}{\partial x}\, dx + \frac{\partial \omega}{\partial y}\, dy \right) \tag{7}$$

and the integral on the right is zero because ζ is zero on the contour C.

A–30. A NECESSARY CONDITION FOR EXTREMA OF A DOUBLE INTEGRAL — AN ANALOGUE OF THE LEGENDRE CONDITION. (*See Section* 22.) *If a smooth function* $\bar{u}(x, y)$ *furnishes the functional*

$$J[u] = \iint_D f(x, y, u, u_x, u_y)\, dx\, dy \tag{1}$$

with a weak extremum, then

$$\bar{f}_{pp} \bar{f}_{qq} - \bar{f}_{pq}^2 \geq 0 \tag{2}$$

at every interior point of the region D. For a minimum it is also necessary that

$$\bar{f}_{pp} \geq 0 \tag{3}$$

---

*We can prove relation (7) in an elementary manner if the mixed partial derivative $\partial^2\omega/\partial x \partial y$ exists and is continuous. This assumption is not necessary but it is sufficient that the first derivatives exist and are continuous. In this connection, see de la Valle-Poussin, *Cours d'analyse infinitésimal*, Vol. 2 (Moscow and Leningrad, 1933), p. 26. A generalization of formula (7) is obtained in a series of works of which we mention two: W. H. Young, "On a new set of conditions for a formula for an area," *Proc. London Math. Soc.*, Vol. 21, 1922; J. Schauder, "Uber die Umkehrung eines Satzes aus der Variationsrechnung," *Acta Szeged*, Vol. 4, 1928, pp. 38–50. In Schauder's work, Haar's lemma was first formulated completely and proved for general assumptions about the region *D*.

and for a maximum it is necessary that

$$\bar{f}_{pp} \leq 0. \tag{3'}$$

PROOF: Suppose that one of the inequalities (2), (3) does not hold at some interior point (x_0, y_0) of the region D. We show that then the function $\bar{u}(x, y)$ will not furnish a minimum. It will follow automatically (if we replace f with $-f$) that the function $\bar{u}(x, y)$ cannot supply the functional with a maximum if either of inequalities (2), (3') does not hold at the point (x_0, y_0).

Our hypothesis and continuity provide that one of the inequalities (2), (3) will not hold in an entire circle K of radius ρ with center (x_0, y_0). But we can find an angle α for which

$$\bar{f}_{pp} \cos^2 \alpha + 2\bar{f}_{pq} \cos \alpha \sin \alpha + \bar{f}_{qq} \sin^2 \alpha \leq -\sigma^2 < 0$$

for an arbitrary point of the circle K. Continuity insures that for an angle α' sufficiently close to α we shall also have everywhere in K at least

$$\bar{f}_{pp} \cos^2 \alpha' + 2\bar{f}_{pq} \cos \alpha' \sin \alpha' + \bar{f}_{qq} \sin^2 \alpha' \leq -\frac{\sigma^2}{2}.$$

We may assume that

$$0 < \alpha - \alpha' \leq \frac{\pi}{2}.$$

Now we introduce the straight lines:

$$L_1(x, y) \equiv \beta - (x - x_0) \cos \alpha' - (y - y_0) \sin \alpha' = 0, \tag{4a}$$

$$L_2(x, y) \equiv \beta - (x - x_0) \cos \alpha - (y - y_0) \sin \alpha = 0, \tag{4b}$$

$$L_3(x, y) \equiv \beta + (x - x_0) \cos \alpha' + (y - y_0) \sin \alpha' = 0, \tag{4c}$$

$$L_4(x, y) \equiv \beta + (x - x_0) \cos \alpha + (y - y_0) \sin \alpha = 0 \tag{4d}$$

in which $\beta > 0$ is so small that the rhombus bounded by these lines lies entirely in K. The diagonals of the rhombus divide it into triangles Δ_i, $(i = 1, 2, 3, 4)$ whose bases are the lines (4a), (4b), (4c), (4d) respectively. We also introduce the piecewise-smooth function $\zeta(x, y)$ by setting

$$\zeta(x, y) = \begin{cases} 0 & \text{outside the rhombus} \\ L_i(x, y) & \text{in the triangle } \Delta_i, \qquad (i = 1, 2, 3, 4) \end{cases}$$

and consider the one-parameter family of admissible functions

$$u = \bar{u}(x, y) + \epsilon\zeta(x, y).$$

Assuming that the function $\bar{u}(x, y)$ furnishes the functional (1) with a minimum, we write

$$\delta J = 0, \qquad \delta^2 J \geq 0.$$

We can establish a contradiction by showing that for sufficiently small $\beta > 0$

$$\delta^2 J < 0. \tag{5}$$

To prove this we write $\delta^2 J$ in the form

$$\delta^2 J = \epsilon^2 \iint (\bar{f}_{uu}\zeta^2 + 2\bar{f}_{up}\zeta\zeta_x + 2\bar{f}_{uq}\zeta\zeta_y)\, dx\, dy$$
$$+ \epsilon^2 \iint (\bar{f}_{pp}\zeta_x^2 + 2\bar{f}_{pq}\zeta_x\zeta_y + \bar{f}_{qq}\zeta_y^2)\, dx\, dy = \epsilon^2(I_1 + I_2)$$

and noticing that

$$0 \leq \zeta \leq \beta, \qquad |\zeta_x| \leq 1, \qquad |\zeta_y| \leq 1$$

we see that

$$|I_1| \leq \beta(\beta + 4)MS$$

in which S is the area of the rhombus and M is the upper bound of the absolute values of the functions $\bar{f}_{uu}, \bar{f}_{up}, \bar{f}_{uq}$. By evaluating I_2 we find that

$$I_2 = \iint\limits_{\Delta_1+\Delta_3} (\bar{f}_{pp}\cos^2\alpha' + 2\bar{f}_{pq}\cos\alpha'\sin\alpha' + \bar{f}_{qq}\sin^2\alpha')\, dx\, dy$$
$$+ \iint\limits_{\Delta_2+\Delta_4} (\bar{f}_{pp}\cos^2\alpha + 2\bar{f}_{pq}\cos\alpha\sin\alpha + \bar{f}_{qq}\sin^2\alpha)\, dx\, dy \leq -\frac{\sigma^2}{2}S.$$

For sufficiently small $\beta > 0$, the value of I_1 will have no influence on the sign of the sum of I_1 and I_2; thus we have proved inequality (5).

Note that the functional (1) and the corresponding calculus of variations problem is called *regular* if for any point of the considered region of xyu-space and for arbitrary finite p, q

$$f_{pp}f_{qq} - f_{pq}^2 > 0.$$

Hilbert first stressed the importance of regular calculus of variations problems for analysis, mathematical physics, and geometry at the First International Congress of Mathematics (Paris, 1900). Among the important results presented there was a problem by Hilbert: "*Are there nonanalytic solutions of regular calculus of variations problems?*"* S. N. Bernstein answered Hilbert's question in his dissertation of 1904 and, with some modifications, in the work "Investigation and integration of differential equations with partial derivatives of elliptic type of the second order," Har'kov, soobs. *Har'k. Mat. Obs.*, Vol. 11, 1907. The fundamental result is Bernstein's theorem.

BERNSTEIN'S THEOREM: *If* $\bar{u}(x, y)$ *is a thrice continuously differentiable function that satisfies the equation*

$$F(x, y, u, p, q, r, s, t) = 0 \tag{6}$$

in which

$$p = \frac{\partial u}{\partial x}, \qquad q = \frac{\partial u}{\partial y}, \qquad r = \frac{\partial^2 u}{\partial x^2}, \qquad s = \frac{\partial^2 u}{\partial x \partial y}, \qquad t = \frac{\partial^2 u}{\partial y^2}$$

and if

$$\frac{\partial \bar{F}}{\partial r}\frac{\partial \bar{F}}{\partial t} - \frac{1}{4}\left(\frac{\partial \bar{F}}{\partial s}\right)^2 > 0 \tag{7}$$

and if the function F is an analytic function of its arguments, then $\bar{u}(x, y)$ *is an analytic function of all its arguments.*† In calculus of variations problems, equation (6) is the Euler equation for the functional (1) and hence has the form

$$f_u - (f_{px} + f_{pu}p + f_{pp}r + f_{pq}s) - (f_{qy} + f_{qu}q + f_{qp}s + f_{qq}t) = 0.$$

In this case the left side of relation (7) is

$$f_{pp}f_{qq} - f_{pq}^2.$$

We introduce a simple example that illustrates the essential role

*Of course, we are speaking of functionals whose integrand functions are analytic.

†See Louis Nizenberg, "On nonlinear elliptic partial differential equations and Hölder continuity," *Comm. Pure Appl. Math. VI*, 1953, New York, for weaker assumptions on $\bar{u}(x, y)$.

played by the regularity of the functional in the analyticity of extremals. Consider the nonregular functional

$$\iint_D (u_x^2 - u_y^2)\, dx\, dy$$

the integrand function of which is analytic. The Euler equation is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

and its general solution is

$$u = \Phi(x + y) + \Psi(x - y)$$

in which Φ and Ψ are arbitrary twice differentiable but nonanalytic functions.

A–31. HAAR'S THEOREM ON REGULAR FUNCTIONALS. (*See Section* 22.) *Suppose the integrand function of the functional*

$$J[u] = \iint_D f(x, y, p, q)\, dx\, dy \tag{1}$$

does not contain u and is regular, that is

$$f_{pp} f_{qq} - f_{pq}^2 > 0$$
$$[(x, y) \in D,\ -\infty < p < \infty,\ -\infty < q < \infty]. \tag{2}$$

Suppose further that

$$f_{pp} > 0 \tag{3}$$

and that in a simply connected region D there exist smooth functions $\bar{u}(x, y)$ *and* $\omega(x, y)$ *for which everywhere in D*

$$\frac{\partial \omega}{\partial x} = \bar{f}_q, \qquad \frac{\partial \omega}{\partial y} = -\bar{f}_p. \tag{4}$$

If these conditions are satisfied, then

$$J[u] > J[\bar{u}]$$

for every smooth function $u(x, y)$ *different from* $\bar{u}(x, y)$ *that coincides with* $\bar{u}(x, y)$ *on the boundary of the region D; that is, for given values*

on the boundary of the region D, the function $\bar{u}(x, y)$ furnishes the functional with an absolute (and moreover strict) minimum.

PROOF: Suppose that $u(x, y)$ is a smooth function in the region D; it is different from $\bar{u}(x, y)$ but coincides with it on the boundary of D. Let us set

$$u(x, y) - \bar{u}(x, y) = \zeta(x, y)$$

so that $\zeta(x, y)$ is a smooth function that vanishes on the boundary of D. Designating by θ an arbitrary number in the interval $[0, 1]$, we introduce the smooth function

$$\tilde{u}(x, y) = (1 - \theta)\bar{u}(x, y) + \theta u(x, y) = \bar{u}(x, y) + \theta\zeta(x, y)$$

and we set

$$\phi(\theta) = J[\tilde{u}].$$

First we make use of the fact that

$$\phi'(0) = 0. \tag{5}$$

Indeed,

$$\phi'(0) = \iint_D \left(\tilde{f}_p \frac{\partial \zeta}{\partial x} + \tilde{f}_q \frac{\partial \zeta}{\partial y}\right) dx\, dy. \tag{6}$$

Condition (4) and the fact that the region D is simply connected provide that for an arbitrary contour γ lying in D

$$\int_\gamma \tilde{f}_q\, dx - \tilde{f}_p\, dy = 0.$$

Hence, if we apply Haar's lemma (see Section 29 of the Appendix) we see that the right side of equation (6) is zero. Now we show that

$$\phi''(\theta) > 0. \tag{7}$$

To do this, consider the expression

$$\begin{aligned}\phi''(\theta) &= \iint_D (\tilde{f}_{pp}\zeta_x^2 + 2\tilde{f}_{pq}\zeta_x\zeta_y + \tilde{f}_{qq}\zeta_y^2)\, dx\, dy \\ &= \iint_D \frac{1}{\tilde{f}_{pp}} \{(\tilde{f}_{pp}\zeta_x + \tilde{f}_{pq}\zeta_y)^2 + (\tilde{f}_{pp}\tilde{f}_{qq} - \tilde{f}_{pq}^2)\zeta_y^2\}\, dx\, dy.\end{aligned}$$

It follows immediately from conditions (2) and (3) that the right member is always greater than zero with the exception of the case in which

$$\zeta_x = \zeta_y = 0$$

everywhere in D. But this is not possible because

$$u(x, y) \not\equiv \bar{u}(x, y)$$

in D and

$$\zeta(x, y) = 0$$

on the boundary of D. It follows from inequality (7) that the curve

$$\sigma = \phi(\theta), \qquad (0 \leq \theta \leq 1)$$

is strictly convex; this means that it lies above its tangent. Considering the tangent at the point $\theta = 0$ we have

$$\phi(\theta) > \phi(0) + \theta\phi'(0), \qquad (0 < \theta \leq 1).$$

This relation and equation (5) imply that

$$\phi(1) > \phi(0) \qquad \text{or} \qquad J[u] > J[\bar{u}]$$

which was to be proved.

A–32. A SPECIAL CASE OF AN ISOPERIMETRIC PROBLEM WITH NATURAL CONDITION AT ONE END POINT. (*See Sections* 25 *and* 31.) We wish to find an extremum for the functional

$$J[y] = \phi[y(a)] + \int_a^b f(x, y, y')\, dx$$

for the condition

$$y(b) = b_1 \tag{1}$$

and side conditions

$$\int_a^b \phi_i(x)y(x)\, dx = l_i, \qquad (i = 1, 2, \cdots, k) \tag{2}$$

in which the $\phi_i(x)$ are given continuous linearly independent functions.

Suppose $y = \bar{y}(x)$ is a smooth curve that furnishes this extremum and lies, with the possible exception of its end points, in a

given region G of the xy-plane. We shall assume that $f(x, y, z)$ and its partial derivatives $f_y(x, y, z), f_z(x, y, z)$ are continuous functions of their arguments for (x, y) in G and arbitrary finite z. We shall also assume that the function $\phi(y)$ is continously differentiable on the intersection of the region G with the straight line $x = a$. We show that the function $\bar{y}(x)$ satisfies the equation

$$\frac{d}{dx} f_{y'} - f_y + \sum_{i=1}^{k} \lambda_i \phi_i = 0 \tag{3}$$

in which the λ_i are constants, and the end condition at the left-hand end point is

$$f_{y'}(a, y(a), y'(a)) = \phi'[y(a)]. \tag{4}$$

Here only the boundary condition (4) is new; the equation (3) can be obtained from the results of Section 25. However, we shall also derive equation (3) by using a method that does not depend on Section 25.

To accomplish this we choose an arbitrary smooth function $\eta(x)$ that satisfies the condition

$$\eta(b) = 0$$

and the side conditions

$$\int_a^b \phi_i(x)\eta(x)\, dx = 0, \qquad (i = 1, 2, \cdots, k) \tag{5}$$

and set

$$y = \bar{y}(x) + \epsilon\eta(x).$$

If we equate the first variation of the functional

$$J[y] = J[\bar{y}(x) + \epsilon\eta(x)]$$

to zero, we obtain the equation

$$\eta(a)\phi'[\bar{y}(a)] + \int_a^b (\bar{f}_y\eta + \bar{f}_{y'}\eta')\, dx = 0 \tag{6}$$

from which we obtain

$$\eta(a)\phi'[\bar{y}(a)] + \int_a^b \eta' \left(\bar{f}_{y'} - \int_a^x \bar{f}_y\, dx \right) dx = 0. \tag{7}$$

First we impose the condition $\eta(a) = 0$ and if we set $\eta'(x) = \zeta(x)$

we may write equation (7) in the form

$$\int_a^b \zeta \left(\bar{f}_{y'} - \int_a^x \bar{f}_y \, dx \right) dx = 0.$$

Because the side conditions (5) provide that

$$\int_a^b \zeta \left\{ \int_a^x \phi_i(x) \, dx \right\} dx = 0, \qquad (i = 1, 2, \cdots, k)$$

and also since

$$\int_a^b \zeta \cdot 1 \cdot dx = 0$$

the lemma in Section 24 of the Appendix implies that

$$\bar{f}_{y'} - \int_a^x \bar{f}_y \, dx = \lambda - \sum_{i=1}^{k} \lambda_i \int_a^x \phi_i(x) \, dx.$$

We can differentiate this equation to obtain

$$\frac{d}{dx} \bar{f}_{y'} - \bar{f}_y + \sum_{i=1}^{k} \lambda_i \phi_i(x) = 0 \tag{8}$$

which shows that $\bar{y}(x)$ satisfies equation (3). If we use the assumption that $\eta(a) = 0$ and substitute the expression for $\bar{f}_y$ obtained from equation (8) in the integrand function of formula (6) and also take the side conditions (5) into account, we find that the boundary condition (4) is satisfied for $\bar{y}(x)$.

A–33. THE STURM-LIOUVILLE PROBLEM. (*See Section* 31.) The Sturm-Liouville problem for a finite interval $[a, b]$ is a series of questions related to the equation

$$L[y] \equiv -\frac{d}{dx} \left\{ p(x) \frac{dy}{dx} \right\} + q(x)y = \lambda r(x)y \tag{1}$$

for the boundary conditions

$$\begin{aligned} &\alpha_1 y'(a) - \alpha_2 y(a) = 0, \qquad (\alpha_1^2 + \alpha_2^2 = 1) \\ &\beta_1 y'(b) + \beta_2 y(b) = 0, \qquad (\beta_1^2 + \beta_2^2 = 1). \end{aligned} \tag{2}$$

Here $p(x) > 0$, $q(x)$, $r(x) > 0$ are given continuous functions in

$[a, b]$, $q(x)$ real; λ is a parameter and α_i, β_i, $(i = 1, 2)$ are given real numbers which we shall consider nonnegative.

The fundamental question concerns the existence of values of λ (we call them *characteristic values*) for which the system (1), (2) has a nontrivial solution (*characteristic function*). This question can be answered with the aid of direct methods of the calculus of variations; we shall answer it in five parts.

A. As an illustration of various boundary conditions we shall take the example $\alpha_1 \neq 0$ and $\beta_1 = 0$; thus condition (2) has the form

$$\begin{aligned} y'(a) - \alpha y(a) &= 0, \qquad (\alpha \geq 0), \\ y(b) &= 0. \end{aligned} \tag{3}$$

We consider the quadratic form

$$J[y] = \alpha p(a)y^2(a) + \int_a^b (py'^2 + qy^2)\, dx \tag{4}$$

and also the bilinear form

$$J[u, v] = \alpha p(a)u(a)v(a) + \int_a^b (pu'v' + quv)\, dx.$$

We can integrate by parts to verify that

$$\int_a^b L[u]v\, dx = J[u, v] \tag{5}$$

if

(1) the functions u, v satisfy the conditions

$$u'(a) - \alpha u(a) = 0, \qquad v(b) = 0$$

(2) the function v is absolutely continuous

(3) we apply the operator L to the function u.

In particular, if $\tilde{y}(x)$ is a characteristic function and $\tilde{\lambda}$ is a characteristic value, formula (5) and $u = v = \tilde{y}(x)$ yield

$$\tilde{\lambda} \int_a^b r(x)\tilde{y}^2\, dx = \int_a^b L[\tilde{y}]\tilde{y}\, dx = J[\tilde{y}, \tilde{y}] = J[\tilde{y}].$$

B. The simplest properties of characteristic functions are the following:

(1) *characteristic functions* $y_1(x)$, $y_2(x)$ *that belong to two different characteristic values* λ_1, λ_2 satisfy the relation

$$(y_1, y_2) \equiv \int_a^b r(x)y_1(x)y_2(x)dx = 0.$$

(2) *to each characteristic value belongs only one characteristic function, unique up to a constant multiplier.*

To prove property (1), consider the identities

$$L[y_1] = \lambda_1 r(x)y_1,$$

$$L[y_2] = \lambda_2 r(x)y_2.$$

These identities and formula (5) which is obviously applicable here provide that

$$\lambda_1(y_1, y_2) = \int_a^b L[y_1]y_2\, dx = J[y_1, y_2],$$

$$\lambda_2(y_1, y_2) = \int_a^b y_1 L[y_2]\, dx = J[y_1, y_2]$$

and hence

$$\lambda_1(y_1, y_2) = \lambda_2(y_1, y_2).$$

This implies that

$$(y_1, y_2) = 0$$

because

$$\lambda_1 \neq \lambda_2.$$

We prove property (2) by assuming the contrary. Suppose that to a characteristic value λ there correspond two linearly independent characteristic functions $y_1(x)$, $y_2(x)$. Then we can find two constants A_1, A_2 not zero simultaneously for which the linear combination

$$u(x) = A_1y_1(x) + A_2y_2(x)$$

is not identically zero and is a characteristic function that vanishes for $x = a$. But then $u(a) = 0, u'(a) = 0$ and a uniqueness theorem from the theory of differential equations provides that if the function $p(x)$ has a continuous derivative, we have obtained a contradiction. However, because we have not assumed that $p(x)$

is differentiable, we can make the transformation

$$t = \int_a^x \frac{dx}{p(x)} \equiv \omega(x)$$

from which we obtain

$$p(x)\frac{du}{dx} = \frac{du}{dt}, \qquad \frac{d}{dx}\left\{p\frac{du}{dx}\right\} = \frac{1}{p(x)}\frac{d^2u}{dt^2}$$

and the equation

$$L[u] = \lambda r(x)u$$

takes the form

$$-\frac{d^2u}{dt^2} + q_1(t)u = \lambda r_1(t)u.$$

The uniqueness theorem applies to an equation of this form.

From property (1) it follows that *there are no complex characteristic values.* If λ were a complex number and $y(x)$ the corresponding characteristic function, then the number conjugate to λ would be a characteristic value, and the function conjugate to $y(x)$ would be a characteristic function. But then property (1) implies

$$\int_a^b r(x)|y(x)|^2\,dx = 0$$

which is absurd since the characteristic function is not identically zero. Thus, as is assumed in the calculus of variations, we may consider all numbers and functions real. We shall call two functions $\phi(x), \psi(x)$ *orthogonal* if $(\phi, \psi) = 0$, and we shall call the function $\phi(x)$ *normalized* if

$$||\phi|| \equiv \sqrt{\int_a^b r(x)[\phi(x)]^2\,dx} = 1.$$

c. We now propose the following calculus of variations problem: Find the absolute minimum of the functional

$$J[y] = \alpha p(a)y^2(a) + \int_a^b (py'^2 + qy^2)\,dx \tag{4}$$

for the boundary condition

$$y(b) = 0 \tag{6}$$

and the side condition

$$\int_a^b r(x)y^2\,dx = 1. \tag{7}$$

Taking some (for example linear) function $y_0(x)$ that satisfies conditions (6) and (7) and computing $J[y_0] = A$, we can restrict ourselves to the consideration of only those admissible functions $y(x)$ for which

$$J[y] \leq A.$$

Then

$$\alpha p(a)y^2(a) + p_{\min}\int_a^b y'^2\,dx \leq A + \int_a^b |q(x)|y^2\,dx \leq A + \left\{\frac{|q|}{r}\right\}_{\max}\int_a^b r(x)y^2\,dx = A + \left\{\frac{|q|}{r}\right\}_{\max}$$

from which we obtain

$$\int_a^b y'^2\,dx \leq \frac{A + \{|q|/r\}_{\max}}{p_{\min}} = A_1^2.$$

On the other hand it follows from condition (7) that

$$\{y^2\}_{\min} \leq \frac{1}{(b-a)r_{\min}} = A_2.$$

If the maximum of the function $y^2(x)$ on the interval $[a, b]$ is attained at the point ξ_2, and the minimum at the point ξ_1, then

$$\{y^2\}_{\max} - \{y^2\}_{\min} = 2\int_{\xi_1}^{\xi_2} yy'\,dx \leq 2|y|_{\max}\sqrt{b-a}\sqrt{\int_a^b y'^2\,dx} \leq 2A_1\sqrt{b-a}\,|y|_{\max}.$$

Hence

$$|y|_{\max}^2 \leq A_2 + 2A_1\sqrt{b-a}\,|y|_{\max}$$

and it follows that

$$|y|_{\max} \leq m$$

in which m depends only on $b - a$, A_1 and A_2. So we can restrict ourselves to the consideration of curves $y = y(x)$ that lie in a rectangle

$$a \leq x \leq b, \qquad -m \leq y \leq m$$

which takes the place of the region G. In this rectangle, the function

$$f(x, y, z) = p(x)z^2 + q(x)y^2$$

satisfies the inequality

$$f(x, y, z) \geq p_{\min}z^2 - m^2|q|_{\max}.$$

The function $f_z(x, y, z)$ is monotonic and we also have the inequality

$$f_{zz}(x, y, z) = 2p(x) \geq 2p_{\min} > 0.$$

Hence, for conditions (6) and (7), the functional $J[y]$ has an absolute minimum. Let us designate the minimizing function by $y_1(x)$ and the minimum value by λ_1. Thus for an arbitrary absolutely continuous function $y(x)$ that satisfies conditions (6) and (7)

$$J[y] \geq J[y_1] = \lambda_1. \tag{8}$$

It follows that the function $y_1(x)$ furnishes an absolute minimum (equal to zero) for the functional

$$J_1[y] = J[y] - \lambda_1 \int_a^b r(x)y^2\,dx$$

for condition (6).

Suppose that there exists an absolutely continuous function $u(x)$ satisfying condition (6) for which

$$J_1[u] < 0.$$

This inequality insures that $u(x)$ is not identically zero so that $||u|| \neq 0$. Hence we can introduce the function

$$\tilde{y}(x) = \frac{u(x)}{||u||}$$

that satisfies conditions (6) and (7) and for which

$$J[\tilde{y}] - \lambda_1 = J_1[\tilde{y}] = \frac{1}{||u||^2} J_1[u] < 0.$$

This inequality contradicts inequality (8) and hence is absurd.

The function $y_1(x)$ furnishes the functional $J_1[y]$ with an absolute minimum; thus the general results of Section 31 provide that $y_1(x)$ is a smooth function and Section 32 of the Appendix insures that $y_1(x)$ satisfies the equation

$$-\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + q(x)y = \lambda_1 r(x)y$$

and also the boundary condition

$$y'(a) - \alpha y(a) = 0.$$

Consequently, the minimal value of the functional $J[y]$ for conditions (6) and (7) is the characteristic value of the problem (1), (3), and the minimizing function $y_1(x)$ is the characteristic function.

For an arbitrary characteristic value $\tilde{\lambda}$ different from λ_1, if it exists, relation (5) provides that the corresponding characteristic function $\tilde{y}(x)$ satisfies

$$\tilde{\lambda} = J[\tilde{y}]$$

and relation (8) provides that

$$J[\tilde{y}] \geq \lambda_1$$

and we know that $\tilde{\lambda} \neq \lambda_1$. Thus

$$\tilde{\lambda} > \lambda_1$$

which implies that the characteristic value λ_1 is the smallest. It is called the *first characteristic value;* this is consistent with the notation.

D. We prove an important property of the first characteristic function: that it does not vanish on the whole interval $[a, b)$ open at the right. Suppose the contrary; designate by c the point of the

half-closed interval at which $y_1(x)$ vanishes and set

$$u(x) = \begin{cases} y_1(x), & (a \leq x \leq c) \\ 0, & (c \leq x \leq b). \end{cases}$$

The function $u(x)$ satisfies condition (6) and is an admissible function for the functional $J_1[y]$. The identity

$$L[y_1] = \lambda_1 r(x) y_1$$

provides that the quantity $J_1(u)$ which satisfies the equation

$$J_1[u] = \alpha p(a) y_1^2(a) + \int_a^c (p y_1'^2 + q y_1^2)\, dx - \lambda_1 \int_a^c r y_1^2\, dx$$

also satisfies the equation

$$J_1[u] = \int_a^c y_1 \{(-p y_1')' + q y_1\}\, dx - \lambda_1 \int_a^c r y_1^2\, dx = 0.$$

Hence the function $u(x)$, $(a \leq x \leq b)$ furnishes the functional $J_1[y]$ with a minimum. This implies that at the point $x = c$ the Weierstrass-Erdmann corner condition must be satisfied. Here this means that $p(c)y_1'(c) = 0$ or $y_1'(c) = 0$; as we have shown, this is incompatible with the fact that $y_1(c) = 0$ because $y_1(x) \not\equiv 0$.

E. To find the second characteristic value we seek an absolute minimum for the functional $J[y]$ for condition (6) and the two side conditions

$$\int_a^b r(x) y^2\, dx = 1, \qquad \int_a^b r(x) y_1(x) y\, dx = 0$$

in which $y_1(x)$ is the first characteristic function.

We establish that this minimum exists as we did in part C. We denote the minimizing function by $y_2(x)$ and the minimum value by λ_2. We can also establish that the function $y_2(x)$ furnishes an absolute and proper minimum for the functional

$$J_2[y] = J[y] - \lambda_2 \int_a^b r(x) y^2\, dx$$

for condition (6) and the side condition

$$\int_a^b r(x) y_1(x) y\, dx = 0.$$

It follows from Section A–32 that $y_2(x)$ satisfies the boundary condition

$$y'(a) - \alpha y(a) = 0$$

and the differential equation

$$L[y] = \lambda_2 r(x)y + \mu_1 r(x)y_1. \tag{9}$$

Relation (5) allows us to write

$$\int_a^b L[y_2]y_1\,dx = \int_a^b L[y_1]y_2\,dx = \lambda_1 \int_a^b r(x)y_1y_2\,dx = 0$$

and if we substitute y_2 for y in equation (9), multiply by y_1, and integrate, we obtain

$$\lambda_2 \int_a^b r(x)y_2y_1\,dx + \mu_1 \int_a^b r(x)y_1^2\,dx = 0.$$

Hence $\mu_1 = 0$. Equation (9) thus becomes

$$L[y] = \lambda_2 r(x)y$$

which implies that λ_2 is the second characteristic value and $y_2(x)$ is the second characteristic function.

To find the third characteristic function we seek the absolute minimum of the functional $J[y]$ for condition (6) and the three side conditions

$$\int_a^b r(x)y^2\,dx = 1,$$

$$\int_a^b r(x)y_1(x)y\,dx = 0,$$

$$\int_a^b r(x)y_2(x)y(x)\,dx = 0.$$

By continuing this process we obtain an infinite sequence of characteristic values

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

and a corresponding set of orthonormal characteristic functions

$$y_1(x), y_2(x), y_3(x), \cdots.$$

A–34. THE SET OF CHARACTERISTIC VALUES OF THE STURM-LIOUVILLE PROBLEM DOES NOT HAVE AN UPPER BOUND. Let us suppose that

$$\lambda_i < A < \infty, \qquad (i = 1, 2, \cdots).$$

Then

$$J[y_i] < A, \qquad (i = 1, 2, 3, \cdots) \tag{1}$$

in which $\{y_i(x)\}_{i=1}^{\infty}$ is the sequence of characteristic functions orthogonal and normalized relative to the weight function $r(x)$. From relation (1) it follows that

$$\int_a^b [y_i'(x)]^2\,dx \leq \frac{A + \{|q|/r\}_{\max}}{p_{\min}} = B^2, \qquad (i' = 1, 2, 3, \cdots)$$

and hence, as in Section 28, it follows that

$$|y_i(x'') - y_i(x')| \leq B\sqrt{x'' - x'},$$
$$(a \leq x' < x'' \leq b, i = 1, 2, 3, \cdots).$$

This implies that the sequence $\{y_i(x)\}_{i=1}^{\infty}$ is uniformly bounded and equicontinuous in the interval $[a, b]$. Thus we can apply a theorem of Arzela that insures the existence of a subsequence $\{y_{n_i}(x)\}_{i=1}^{\infty}$ which converges uniformly in the interval $[a, b]$. This theorem provides that

$$\lim_{i,j\to\infty} \int_a^b r(x)[y_{n_j}(x) - y_{n_i}(x)]^2\,dx = 0$$

which is absurd because for $n_i \neq n_j$

$$\int_a^b r(x)[y_{n_j}(x) - y_{n_i}(x)]^2\,dx = \int_a^b r(x)[y_{n_j}(x)]^2\,dx$$
$$- 2\int_a^b r(x)y_{n_i}(x)y_{n_j}(x)\,dx + \int_a^b r(x)[y_{n_i}(x)]^2\,dx = 2.$$

A–35. THE CLOSURE THEOREM OF V. A. STEKLOV. Every orthogonal system of functions generates a series similar to a trigonometric Fourier series. The system of characteristic functions

$$y_1(x), y_2(x), \cdots \tag{1}$$

of a Sturm-Liouville problem thus generates such a series.* That is to say, every summable function $f(x)$, $(a \leq x \leq b)$ can be expressed as a series in the functions (1)

$$f(x) \sim \sum_{i=1}^{\infty} a_i y_i(x).$$

Here

$$a_i = \int_a^b r(t)f(t)y_i(t)\,dt = (f, y_i), \qquad (i = 1, 2, 3, \cdots). \quad (2)$$

V. A. Steklov first proved that *if the square of the absolute value of* $f(x)$ *is integrable then*

$$\int_a^b r(x)|f(x)|^2\,dx = \sum_{i=1}^{\infty} |a_i|^2. \quad (3)$$

For Fourier series, this relation is known as the *Parseval-Lyapunov equation* and Steklov called it the *closure equation*. It is sufficient to prove it for real functions. We shall also assume that $f(x)$ is an absolutely continuous function that vanishes at $x = b$.†

To prove Steklov's theorem we must use the fact that the sequence of characteristic values is not bounded from above. Suppose that $f(x)$ is an absolutely continuous function equal to zero for $x = b$. We set

$$f(x) = \sum_{i=1}^{n} a_i y_i(x) + R_n(x) \quad (4)$$

in which the coefficients a_i are defined by the system (2).

---

*We shall assume that the discussion concerns the Sturm-Liouville problem for the equation and boundary conditions considered in Section 33 of the Appendix. In particular, we assume the condition at the right-hand end point to be $y(b) = 0$.

†The general case of an arbitrary function $f(x)$ the square of whose absolute value is integrable reduces to this particular case. This is a consequence of the fact that for arbitrary $\epsilon > 0$ an absolutely continuous function $f_\epsilon(x)$ (even a polynomial) can be found which is equal to zero for $x = b$ and which satisfies the inequality

$$\int_a^b r(x)|f(x) - f_\epsilon(x)|^2\,dx < \epsilon.$$

Because all the functions $y_i(x)$ and also $f(x)$ are zero at $x = b$, it follows that

$$R_n(b) = 0.$$

As a further property of the functions $R_n(x)$ we have the equations

$$(R_n, y_i) = 0, \qquad (i = 1, 2, \cdots, n)$$

which result from the definition of the coefficients a_i. If we set

$$u(x) = \frac{R_n(x)}{||R_n||}$$

we obtain a function for which

$$u(b) = 0,$$

$$(u, y_i) = 0, \qquad (i = 1, 2, \cdots, n),$$

$$\int_a^b r(x)u^2(x)\,dx = 1.$$

Thus $u(x)$ is an admissible function for finding the $n + 1$st characteristic value by a method of variations. Consequently,

$$J[u] \geq \lambda_{n+1}$$

whence

$$J[R_n] \geq \lambda_{n+1}(R_n, R_n). \tag{5}$$

We note that the equation

$$(f, f) = \sum_{i=1}^{n} a_i^2 + (R_n, R_n) \tag{6}$$

follows from formula (4). On the basis of formula (4) we may write

$$\begin{aligned}(R_n, R_n) &= (f, R_n) = (f, f - \sum_{i=1}^{n} a_i y_i) \\ &= (f, f) - \sum_{i=1}^{n} a_i(f, y_i) = (f, f) - \sum_{i=1}^{n} a_i^2.\end{aligned}$$

Formula (6) shows that proof of the closure equation reduces to proof of

$$\lim_{n \to \infty} (R_n, R_n) = 0.$$

Because

$$\lim_{n\to\infty} \lambda_n = \infty$$

it is sufficient, as inequality (5) demonstrates, to establish that the quantity $J[R_n]$ is bounded from above as $n \to \infty$.

To prove this we write the following equation which is a consequence of relation (4)

$$J[R_n] = J[f] + J\left[\sum_{i=1}^{n} a_i y_i\right] - 2J\left[f, \sum_{i=1}^{n} a_i y_i\right].$$

Both the quadratic functional and the bilinear functional introduced in Section A–33 appear on the right side of this equation. From equation (5) of this section we obtain

$$J\left[\sum_{i=1}^{n} a_i y_i, f\right] = \sum_{i=1}^{n} a_i J[y_i, f] = \sum_{i=1}^{n} a_i \int_a^b L[y_i] f \, dx$$

$$= \sum_{i=1}^{n} a_i \int_a^b \lambda_i r(x) y_i(x) f(x) \, dx = \sum_{i=1}^{n} \lambda_i a_i^2$$

and

$$J\left[\sum_{i=1}^{n} a_i y_i\right] = \sum_{i,k=1}^{n} a_i a_k J[y_i, y_k]$$

$$= \sum_{i,k=1}^{n} \lambda_i a_i a_k \int_a^b r(x) y_i(x) y_k(x) \, dx = \sum_{i=1}^{n} \lambda_i a_i^2.$$

Consequently

$$J[R_n] = J[f] - \sum_{i=1}^{n} \lambda_i a_i^2. \tag{7}$$

Because $\lambda_n \to \infty$, all λ_n from a certain point on are positive. Formula (7) implies that the quantities $J[R_n]$ decrease from a certain point on. Hence the quantities $J[R_n]$ have a finite upper bound; this proves the theorem.

A–36. MAXIMAL-MINIMAL PROPERTY OF CHARACTERISTIC VALUES. In the development of Steklov's theorem (see Section A–33) the nth characteristic value and its corresponding characteristic function were defined by a calculus of variations problem in the

formulation of which the $n-1$st characteristic function appeared. The nth characteristic value can also be defined as an independent extremal property. Fischer first indicated such a property of characteristic values for quadratic forms in a finite number of variables. R. Courant applied it to calculus of variations problems and also constructed and extended a far-reaching theory based on it.*

We introduce a theorem that applies to the problem considered in Section A–33.

THEOREM: *Suppose* $d\{v_1, v_2, \cdots, v_{n-1}\}$ *is the greatest lower bound of the functional* $J[y]$ *in the class of all piecewise-smooth functions* $y(x)$, $(a \leq x \leq b)$ *that satisfy the boundary condition*

$$y(b) = 0 \tag{1}$$

and the conditions

$$\begin{aligned} \int_a^b r(x)y^2\,dx &= 1, \\ \int_a^b r(x)v_i(x)y\,dx &= 0, \qquad (i = 1, 2, \cdots, n-1) \end{aligned} \tag{2}$$

in which $v_1(x), v_2(x), \cdots, v_{n-1}(x)$ *is a system of piecewise-continuous functions in the interval* $[a, b]$. *Then the nth characteristic value is the maximum value* $d\{v_1, v_2, \cdots, v_{n-1}\}$ *in the class of all possible systems of functions* $v_i(x)$, $(i = 1, 2, \cdots, n-1)$

$$\lambda_n = \max_{v_i} d\{v_1, v_2, \cdots, v_{n-1}\}.$$

PROOF: The inequality

$$\lambda_n \leq \max_{v_i} d\{v_1, v_2, \cdots, v_{n-1}\}$$

follows from the fact that

$$d\{y_1, y_2, \cdots, y_{n-1}\} = \lambda_n.$$

Hence we must show that

$$d\{v_1, v_2, \cdots, v_{n-1}\} \leq \lambda_n$$

*See R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1 (Interscience Publishers, Inc., New York, 1953). References to other works may be found there.

for any functions $v_1(x), v_2(x), \cdots, v_{n-1}(x)$. Assuming that these functions have been chosen, we consider a linear combination of the first n characteristic functions $y_i(x)$ that we suppose are normalized

$$y = c_1y_1(x) + \cdots + c_ny_n(x).$$

Condition (1) is satisfied for arbitrary c_i. To satisfy condition (2), we must determine the constants c_i from the equations

$$c_1^2 + c_2^2 + \cdots + c_n^2 = 1 \tag{3}$$

and

$$\int_a^b r(x)v_i(x)[c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)]\,dx = 0, \qquad (i = 1, 2, \cdots, n-1). \tag{4}$$

We can determine these constants by finding a nontrivial solution of the homogeneous equations (4) and choosing the arbitrary factor so that equation (3) is satisfied.

If we use the method applied in Sections A–32 and A–35, we obtain

$$J[y] = \sum_{i=1}^{n} \lambda_i c_i^2.$$

It follows that

$$J[y] \leq \lambda_n \sum_{i=1}^{n} c_i^2 = \lambda_n$$

because

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n.$$

A–37. SOME REMARKS CONCERNING PROBLEMS FOR AN EXTREMUM WITH A SIDE CONDITION. (*See Sections* 25 *and* 26.)

A. Euler first called attention to the reciprocity law for isoperimetric problems. The simplest case of this law is: *The collection of curves that furnish a stationary value for the functional*

$$J[y] = \int_a^b f(x, y, y')\,dx$$

for the condition that the functional

$$K[y] = \int_a^b g(x, y, y')\, dx$$

have a given value coincides with the collection of curves that furnish the functional $K[y]$ with a stationary value for the condition that the functional $J[y]$ have a fixed value.

B. The isoperimetric problem can be reduced to a Lagrange problem. Suppose, for example, that we wish to find an extremum for the functional $J[y]$ for the condition

$$K[y] = 1$$

and such that

$$y(a) = a_1, \qquad y(b) = b_1.$$

We set

$$z = \int_a^x g(x, y, y')\, dx$$

thus

$$z' = g(x, y, y').$$

Now our problem is to find an extremum for the functional $J[y]$ for the condition

$$z' = g(x, y, y')$$

and such that

$$y(a) = a_1, \qquad y(b) = b_1, \qquad z(a) = 0, \qquad z(b) = 1.$$

This is a Lagrange problem.

C. The so-called *Mayer problem* reduces to a Lagrange problem. The problem is this: Considering all possible systems of smooth functions

$$y_0(x), y_1(x), \cdots, y_n(x)$$

that satisfy the differential equations

$$\phi_i(x, y_0, y_1, \cdots, y_n, y_0', y_1', \cdots, y_n') = 0, \qquad (i = 0, 1, 2, \cdots, m < n) \quad (1)$$

and the boundary conditions

$$y_0(a) = a_0, \qquad y_1(a) = a_1, \qquad y_2(a) = a_2, \cdots, y_n(a) = a_n$$
$$y_1(b) = b_1, \qquad y_2(b) = b_2, \cdots, y_n(b) = b_n \qquad (2)$$

we wish to find the system for which the function $y_0(x)$ has a maximum or a minimum at $x = b$.

This problem is equivalent to the problem of finding an extremum for the functional

$$\int_a^b y_0' \, dx$$

for conditions (1) and for boundary conditions (2); that is, it is a particular case of a Lagrange problem.

Every Lagrange problem for the functional

$$J[y] = \int_a^b f(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n') \, dx$$

can be reduced to a Mayer problem. This can immediately be verified by introducing another unknown function

$$y_0 = \int_a^x f(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n') \, dx.$$

MISCELLANEOUS EXERCISES

1. Find extremals for

$$\int_{x_0}^{x_1} (y^2 - y'^2 - 2y \cosh x) \, dx.$$

2. Find extremals for

$$\int_{x_0}^{x_1} (y^2 + y'^2 + 2ye^x) \, dx.$$

3. Find extremals for

$$\int_{x_0}^{x_1} (y^2 - y'^2 - 2y \sin x) \, dx.$$

4. Find extremals for

$$\int_{x_0}^{x_1} (y^2 + 2y'^2 + y''^2) \, dx.$$

5. Find the Euler (Lagrange-Ostrogradsky) equation for

$$\iiint_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + 2uf(x, y, z)\right] dx\, dy\, dz.$$

6. Find a solution with one corner point for the problem of minimizing

$$\int_0^4 (y' - 1)^2(y' + 1)^2\, dx; \qquad y(0) = 0, \qquad y(4) = 2.$$

7. Show that there are no solutions with corner points for the problem of finding extrema for the functionals:

(a) $\displaystyle\int_{x_0}^{x_1} (y'^2 + 2xy - y^2)\, dx; \qquad y(x_0) = y_0, \qquad y(x_1) = y_1$

(b) $\displaystyle\int_0^{x_1} y'^3\, dx; \qquad y(0) = 0, \qquad y(x_1) = y_1.$

8. Are there solutions with corner points for the problem of finding extrema for the functional

$$\int_0^{x_1} (y'^4 - 6y'^2)\, dx; \qquad y(0) = 0, \qquad y(x_1) = y_1?$$

9. Using the fundamental necessary condition $\delta J = 0$, find the function for which the following functional may have an extremum

$$J[y] = \int_0^1 (y''^2 - 2xy)\, dx; \qquad y(0) = y'(0) = 0, \qquad y(1) = \tfrac{1}{120}$$

if $y'(1)$ is not prescribed.

10. Find curves that may furnish an extremum for the functional

$$\int_0^{10} y'^3\, dx; \qquad y(0) = 0, \qquad y(10) = 0$$

subject to the condition that admissible curves cannot pass inside the circle $(x - 5)^2 + y^2 = 9$.

11. Try to get from a point $A(x_0, y_0)$ not on the curve $y = \phi(x)$ to a point $C(x_0, y_0)$ on $y = \phi(x)$ in the shortest time. The velocity of

the motion not on the curve $y = \phi(x)$ is equal to v_1; the velocity along the curve $y = \phi(x)$ is also constant and equal to v_2, $v_2 > v_1$.

12. Show that there is no extremum in the class of continuous functions for

$$\int_{-1}^{2} y'(1 + x^2y')\,dx; \qquad y(-1) = 1, \qquad y(2) = 4.$$

13. Show that $y = 1$ furnishes a strong relative minimum for

$$\int_{-1}^{2} y'(1 + x^2y')\,dx; \qquad y(-1) = y(2) = 1.$$

14. Show that $y = x^3$ furnishes a strong relative minimum for

$$\int_{1}^{2} (x^2y'^2 + 12y^2)\,dx; \qquad y(1) = 1, \qquad y(2) = 8.$$

15. Show that the family of extremals

$$y = \sinh\,(C_1x + C_2)$$

in which C_1 and C_2 are defined by

$$y_0 = \sinh\,(C_1x_0 + C_2), \qquad y_1 = \sinh\,(C_1x_1 + C_2)$$

furnishes a weak relative minimum for

$$\int_{x_0}^{x_1} \frac{1 + y^2}{y'^2}\,dx; \qquad y(x_0) = y_0, \qquad y(x_1) = y_1.$$

16. Show that $y = e^{2x}/3$ furnishes a strong relative minimum for

$$\int_{0}^{1} (y'^2 + y^2 + 2ye^{2x})\,dx; \qquad y(0) = \frac{1}{3}, \qquad y(1) = \frac{e^2}{3}.$$

17. Show that $y = \sin 2x$ furnishes a strong relative maximum for

$$\int_{0}^{\pi/4} (y^2 - y'^2 + 6y \sin 2x)\,dx; \qquad y(0) = 0, \qquad y\left(\frac{\pi}{4}\right) = 1.$$

18. Show that $y = y_1x/x_1$ furnishes a weak relative minimum for

$$\int_{0}^{x_1} \frac{dx}{y'}; \qquad y(0) = 0, \qquad y(x_1) = y_1, \qquad x_1 > 0, \qquad y_1 > 0.$$

19. Show that $y = y_1 x/x_1$ furnishes a weak relative minimum for

$$\int_0^{x_1} \frac{dx}{y'^2}; \qquad y(0) = 0, \qquad y(x_1) = y_1, \qquad x_1 > 0, \qquad y_1 > 0.$$

20. Show that $y = x^3 - 1$ furnishes a strong relative minimum for

$$\int_1^3 (12xy + y'^2)\, dx; \qquad y(1) = 0, \qquad y(3) = 26.$$

21. Find geodesic lines on the circular cylinder $r = R$. It is convenient to use cylindrical coordinates r, ϕ, z.

22. Find extremals for the functional

$$\int_{x_0}^{x_1} y'^2\, dx$$

subject to the condition

$$\int_{x_0}^{x_1} y\, dx = a$$

in which a is a constant.

23. Using the method of Ritz, find an approximate solution to the problem of minimizing

$$\int_0^2 (y'^2 + y^2 + 2xy)\, dx; \qquad y(0) = y(2) = 0$$

and compare it with the exact solution

$$y = \frac{2 \sinh x}{\sinh 2} - x.$$

24. (*See Section* A–5.) Given

$$y'' = 6x$$

find a functional for which it is the Euler equation.

25. Consider a particle lying in a plane acted on by a force directed toward the origin and varying in magnitude inversely as the square of the distance from the origin. Show by means of Hamilton's Principle (Section A–20) that one obtains one of Kepler's laws of planetary motion: Equal areas are swept out in equal times by the radius vector from the origin to the particle.

ANSWERS

This section contains answers to all exercises having numerical solutions, and hints to aid in the solution of others.

CHAPTER 1

1. $(x - C_1)^2 + y^2 = C_2^2$.
2. $y - C_1 = \frac{1}{4}\frac{(x - C_2)^2}{C_1}$
3. $y = \frac{C_1}{x} + C_2$.
4. $y = C_1 \sin(4x - C_2)$.
5. $y = -\frac{x^2}{4} + C_1 x + C_2$.
6. $y = \sinh(C_1 x + C_2)$.
7. $y = C_1 e^x + C_2 e^{-x} + \frac{1}{2}\sin x$.
8. $y = (C_1 x + C_2)\cos x + (C_3 x + C_4)\sin x, \qquad z = 2y + y''$.
9. $y = C_1 x + C_2, \qquad z = C_3 x + C_4$.
10. $y = C_1 x^4 + C_2$.
11. The integral is independent of the path of integration. The calculus of variations problem thus loses its meaning.
12. No continuous function will furnish an extremum.

CHAPTER 3

1. $y \equiv 0$.
2. $y = \pm\sqrt{8x - x^2}$.
3. $\frac{\phi' - y'}{1 + y'\phi'} = 1$. The extremals must intersect at an angle of $\frac{1}{4}\pi$ the curve $y_1 = \phi(x_1)$ on which the end point varies.

4. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0.$

5. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x.$

6. $y = \dfrac{x^7}{7!} + C_1 x^5 + C_2 x^4 + C_3 x^3 + C_4 x^2 + C_5 x + C_6.$

7. $y = \pm 2 \sin n\pi x$ where n is an integer.

8. $y = -\dfrac{5x^2}{2} + \dfrac{7x}{2}, \qquad z = x.$

CHAPTER 4

1. An approximate solution can be written in the form

$$y_n = x(1 - x)(\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n).$$

Carry out the calculation for $n = 0$ and $n = 1$.

2. An approximate solution can be expressed in the form

$$y_n(x) = (x - 1)^2(\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n).$$

Carry out the calculation for $n = 1$. The answer is

$$y = (x - 1)^2(.124 + .218\, x).$$

MISCELLANEOUS EXERCISES

1. $y = \frac{1}{2} \cosh x + C_1 \cos x + C_2 \sin x.$

2. $y = \frac{1}{2} x e^x + C_1 e^x + C_2 e^{-x}.$

3. $y = -\frac{1}{2} x \cos x + C_1 \cos x + C_2 \sin x.$

4. $y = (C_1 x + C_2)e^x + (C_3 x + C_4)e^{-x}.$

5. $f(x, y, z) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}.$

6. There are two broken lines that furnish this functional with an absolute minimum:

$$y = -x, \qquad (0 \leq x \leq 1); \qquad y = x - 2, \qquad (1 \leq x \leq 4)$$

and

$$y = x, \qquad (0 \leq x \leq 3); \qquad y = -x + 6, \qquad (3 \leq x \leq 4).$$

8. A broken line that consists of segments with slopes $\sqrt{3}$ and $-\sqrt{3}$ and passes through the given boundary points.

9. $y = \dfrac{x^5}{120} + \dfrac{(x^2 - x^3)}{24}$.

10. $y = \pm \dfrac{3x}{4}, \qquad \left(0 \leq x \leq \dfrac{16}{5}\right); \qquad y = \pm \sqrt{9 - (x - 5)^2}$

$\left(\dfrac{16}{5} \leq x \leq \dfrac{34}{5}\right); \qquad y = \mp \dfrac{3(x - 10)}{4}, \qquad \left(\dfrac{34}{5} \leq x \leq 10\right)$.

These are curves consisting of a segment of a line tangent to a circle, an arc of a circle, and another line segment tangent to a circle.

11. From the point A one must go to the curve $y = \phi(x)$ along the straight line for which $\cos(\beta - \alpha) = v_1/v_2$; here β is the inclination angle of the line with the x-axis and α is the inclination angle of the tangent to the curve $y = \phi(x)$ at the point of intersection $B(x_1, y_1)$ of the line and the curve.

21. $y = C_1 + C_2 z, \qquad r = R.$

22. $y = \lambda x^2 + C_1 x + C_2$, where C_1, C_2, and λ can be determined from the boundary conditions and the isoperimetric condition.

23. See example 1, Chapter 4.

24. Take $\Phi = (3x^2 - y')^2, \qquad \alpha(x, y) = 12xy - 9x^2, \qquad \beta(x, y) = 6x^2$ to obtain

$$\int (y'^2 + 12xy)\, dx.$$

BIBLIOGRAPHY

Bliss, G. A., *Calculus of Variations*, The Open Court Publishing Company, LaSalle, Illinois, 1935.

Bliss, G. A., *Lectures on the Calculus of Variations*, University of Chicago Press, Chicago, 1946.

Bolza, O., *Lectures on the Calculus of Variations*, University of Chicago Press, Chicago, 1946. Bolza, O., *Vorlesungen über Variationsrechnung*, B. G. Teubner, Leipzig and Berlin, 1909.

Caratheodory, C., *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*, B. G. Teubner, Leipzig and Berlin, 1935.

El'sgol'c, L. E., *Variacionnoye Iscisleniye*, Moscow, 1958.

Forsyth, A. R., *Calculus of Variations*, Dover Publications, New York, 1960.

Fox, C., *An Introduction to the Calculus of Variations*, Oxford University Press, London, 1950.

Hadamard, J. S., *Leçons sur le calcul des variations*, A. Hermann & Fils, Paris, 1910.

Kneser, A., *Lehrbuch der Variationsrechnung*, F. Vieweg & Sohn, Braunschweig, Germany, 1900 and 1925.

Lavrent'ev, M. A., and Lyusternik, L. A., *Osnovy Variacionnogo Iscisleniya*, Moscow and Leningrad, 1935.

Lecat, "Calcul des variations," *Encyclopédie des Sciences Mathématiques* 2, 31, 1913.

Tonelli, L., *Fondamenti di Calcolo delle Variazioni*, N. Zanichelli, Bologna, 1921 and 1923.

University of Chicago Department of Mathematics, *Contributions to the Calculus of Variations*, University of Chicago Press, Chicago, 1930–1932.

Weinstock, R., *Calculus of Variations, with Applications to Physics and Engineering*, McGraw-Hill Book Co., New York, Toronto, London, 1952.

For works prior to 1913, Lecat has a nearly complete bibliography. For literature of the same period, Bliss also included a list in his paper, "Some recent developments in the calculus of variations," *Bull. Am. Math. Soc.* **26**, 1920, pp. 343–361. Caratheodory's book contains an extensive list of works. Bliss includes a long list of books and papers with particular emphasis on the problem of Bolza. References can also be found in *Contributions to the Calculus of Variations.*

INDEX

Absolute minimum:
  for ordinary problem, 131
  for parametric problem, 148
  theorem of Tonelli, 137
Admissible curve, 3
Admissible surface, 100

Bernouilli, John, 4
Bernstein theorem on a global solution, 30
  applications of, 163
Bernstein theorem on analytic solutions, 214
Bliss, G. A., 109
Brachistochrone, 4, 180

Canonical system, 85
Canonical variables, 25, 162
Catenary, 174
Characteristic functions, 220
  normalized, 222
  orthogonal, 222
Characteristic points, 80
Characteristic values, 220, 225, 226, 227
  maximal-minimal property of, 231
Chaplygin's problem, 206
Clairaut theorem on geodesics, 184
Closure equation, 229
Conjugate points, 76, 81
Convergence:
  strong, 157
  weak, 135
Cycloid, 181

Dirichlet integral, 194, 195, 196
Dirichlet principle, 128
Dirichlet problem, 127
Direct method, 127
  steps of, 130
Discontinuous problem, 98
Discontinuous solution for surface of revolution, 176
Double integral problem, 99
  first variation, 101
  Haar theorems, 208, 210, 215, 216
  necessary condition, 211
  second-order derivatives, 108

Element:
  nonsingular, 22
  parametric problem, 45
Energy integral, 187
Envelope, 80, 81
Euler equations, 10, 108, 139
  canonical form, 25
  double integrals, 100, 109, 208
  invariance of, 14
Euler-Weierstrass equation, 43
Extrema:
  absolute, 5
  relative, 5, 6
  conditioned, 109
Extremal, 13, 43
  broken, 13, 191
  existence of, 22
  imbedded in a field, 58

Field:
  for ordinary problem, 47
  for parametric problem, 54
  slope function of, 47, 54
Function space, 3
Functional:
  accessory, 70
  nonsingular, 22
  quasiregular, 45, 62

regular, 45, 62, 213
stationary, 93
variation of, 92, 93

Geodesic, 87, 182, 184, 204
Gram determinant, 109, 111, 198

Haar lemma, 210
Haar lemma on regular functionals, 215
Haar theorem, 208
Hamilton-Jacobi equation, 82, 84, 162, 167
Hamilton-Ostrogradski principle of stationary action, 186
Harmonic functions, 194
Hilbert integral, 47, 81, 87
Hilbert theorem:
for ordinary problem, 21
for parametric problem, 44
on existence of limit curve, 150

Imbedding theorem, 78
Indicatrix, 189
Integrand with derivatives of higher order, 104, 108
Inverse problem, 164
Isoperimetric problem, 113, 200, 201, 217
as Lagrange problem, 234

Jacobi condition, 69, 77, 78, 169
Jacobi equation, 75

Kelvin's problem, 199

Lagrange action, 188
Lagrange bracket, 52
Lagrange equations of the second kind, 189
Lagrange function, 186
Lagrange lemma, 20
Lagrange problem, 117, 234
holonomic conditions for, 118
nonholonomic conditions for, 118, 206
Lagrange theorem for a Mayer family, 53
Laplace equation, 128
Legendre condition, 59, 63, 68, 70, 171
for double integral, 211
Legendre equation, 74, 75
Lemma:
fundamental (Lagrange), 20
for higher derivatives, 106
generalized, 197
for multiple integrals, 102, 104
Liouville surface, 182
Logarithmic potential, 194

Maupertuis-Lagrange principle, 187, 188
Mayer family, 52
Mayer problem, 234
as Lagrange problem, 234
Mechanics, 186
Meusnier, 103
Minimal surfaces, 103
Minimizing sequence, construction of, 143, 144
Minimum, nonexistence of, 168
Multiplier rule:
elementary, 109
for isoperimetric problem, 114
for Lagrange problem, 120

Necessary condition, 8, 11
consequences of, 12, 13
for parametric problem, 42
Jacobi, 78
Legendre, 68
Weierstrass, 65.
Neighborhood:
strong, 5, 40
weak, 6, 41

Ostrogradski, M. V., 103

Parabola of safety, 179
Parameters, dependence of solutions on, 26
Parametric problem, 37
Euler-Weierstrass equation for, 43
field for, 54
properties of integrand for, 39, 40
Parseval-Lyapunov equation, 229
Piecewise smooth, 3

Quasi-regular functional, 45, 62

Regular functional, 45, 62, 213, 215

Riemann, 128
Ritz, minimizing sequence, 143
Roussel, continuity of line integral, 151

Slope function, 47, 54
Steklov closure theorem, 228
Sturm-Liouville problem, 219
  characteristic values of, 228
Sufficient conditions, 59, 63, 172, 173
Surface of revolution, 173, 183

Tonelli theorem on existence of an absolute minimum, 137
  on lower semicontinuity, 150
Trajectory of a field, 37, 48, 54
Transversality relation, 89, 96

Variable end point problem, 93, 94
  natural condition, 97
Variables, canonical, 25, 162
Variations, 92, 93
  for higher derivatives, 104
  for double integrals, 101
  unconstrained, 114, 123

Weierstrass *E*-function, 56, 57, 169
Weierstrass condition, 59, 65, 171
Weierstrass-Erdmann corner condition, 12, 43

ABOUT THE AUTHOR

Professor N. I. Akhiezer was born March 6, 1901. His university instruction was obtained at Kiev. In 1928 he completed his candidacy there and began to teach in high school. In 1933 he was appointed a professor at Kharkov University. Professor Akhiezer's main work in mathematics is concerned with the theory of functions. He is the author of several textbooks and monographs; among these are "Lectures on the Theory of Approximation" and "The Classical Problem of Moments."

ABOUT THE TRANSLATOR

Aline H. Frink received her B.A. degree from Mt. Holyoke College in 1924 after which she taught algebra and physics at the Cobleskill High School in New York. In 1927 she received her M.A. in mathematics from the University of Chicago. She did her graduate work in mathematics at Bryn Mawr College during 1927-28 and returned to the University of Chicago where she received her Ph.D. in 1930. After a year as instructor in mathematics at Mt. Holyoke College, Mrs. Frink joined the Pennsylvania State University where she is now an Associate Professor of mathematics.

A NOTE ON THE PRODUCTION AND DESIGN

The text of this book is set in a monotype face called Bruce Old Style No. 31. It is a compact close-fitting letter that permits more copy in a given space than the average transitional or old style type. It was adapted by Sol Hess from the Old Style Series No. 20, originally cut by the George Bruce's Son & Company. In the first quarter of the nineteenth century, George Bruce first formulated the modern system whereby body sizes of type increase by arithmetical progression. This system, which De Vinne called ingenious and scientific, was first adopted by Bruce's own foundry.

The book was composed by *Technical Composition*, Boston, Massachusetts, printed by *Halliday Lithograph Corporation*, West Hanover, Massachusetts, and bound by *The Book Press*, Brattleboro, Vermont. The paper was manufactured by *P. H. Glatfelter Company*, Spring Grove, Pennsylvania. Henri A. Fluchere of *Art-Tech Services Company*, Irvington-on-Hudson, New York drew the illustrations. Andor Braun of New York City designed the book.

ABOUT THE AUTHOR

PROFESSOR N. I. AKHIEZER was born March 6, 1901. His university instruction was obtained at Kiev. In 1928 he completed his candidacy there and began to teach in high school. In 1933 he was appointed a professor at Kharkov University. Professor Akhiezer's main work in mathematics is concerned with the theory of functions. He is the author of several textbooks and monographs; among these are "Lectures on the Theory of Approximation" and "The Classical Problem of Moments."

ABOUT THE TRANSLATOR

ALINE H. FRINK received her B.A. degree from Mt. Holyoke College in 1924 after which she taught algebra and physics at the Cobleskill High School in New York. In 1927 she received her M.A. in mathematics from the University of Chicago. She did her graduate work in mathematics at Bryn Mawr College during 1927-28 and returned to the University of Chicago where she received her Ph.D. in 1930. After a year as instructor in mathematics at Mt. Holyoke College, Mrs. Frink joined the Pennsylvania State University where she is now an Associate Professor of mathematics.

Jacket design by Stan Phillips